Final Report
Contract N140(70024)70754B
December 1961

• applied • mathematics • •

Mathematical Methods in Acoustic Transducer Array Theory

by

N. G. Parke III

P. M. L. Staff

for

U. S. Navy Underwater Sound Laboratory

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Introduction

This report is a summary of the results of the research carried on during the year under contract No. N140(70024)70754B. The purpose of this work was to attempt the development of mathematical techniques for the determination of pressure maxima in acoustic transducer arrays. In general two different methods of attack were suggested:

(1) the study of the problem using the techniques of generalized harmonic analysis, moment generating functions, and expansions in terms of special functions, and

(2) the development of an analog technique for evaluating the integral representation of the solution.

The report is divided as follows:

Part I The techniques of integral transforms are used to develop an integral representation of the velocity potential for pistons of arbitrary shapes. Specific attention is given to the circular piston with the result that the expression for the velocity potential is simplified. The result is then compared with that obtained by Louis V. King as published in the Canadian Journal of Research.

Part II The methods of Part I are applied to a rectangular piston and a tentative solution for the velocity potential is obtained which is analogous to King's result for the circular disk.

Part III A brief discussion of an alternative formulation of the problem is presented and an expression is derived which relates the velocity potential at a given fixed point to the potential at its projection on the (xy) plane.

Part IV The possibility of analog computations with the maximum use of standard analog equipment is investigated. Here the problem is restricted to an arbitrary array in an infinite plane baffle and to pressures on the surface of this plane. Included are numerous graphs and pressure contours obtained by digital computation. These results are compared with the computations of Sherman and Kass as found in U. S. L. Report No. 495 and will also serve later as a check for the new analog approaches.

Part I

The Basic Equations

We start with the fact that the velocity potential resulting from any source distribution on an infinite, rigid plane S, is given by Rayleigh, J.W.S., The Theory of Sound, Vol. II, MacMillan and Co., London, 1940, p.107.

(1) $\psi = -1/(4\pi)\int_s \partial\psi/\partial n \; e^{-ikr}/r \; ds$

ψ .=. velocity potential

S .=. integration of entire plane S

r .=. distance, source point to field point

(2) $w = \partial\psi/\partial n$.=. boundary condition, i.e., normal velocity distribution on the plane

(3) $p = -i\rho\omega\psi$.=. acoustic pressure

e^{-ikr}/r .=. Green's Function for scalar Helmholtz equation for the plane.

To avoid carrying along excess symbolic baggage we shall discuss the nature of ψ on the plane S.

Equation (1) becomes

(4) $\psi(\xi,\eta,\zeta;k) = -1/(4\pi)\iint_{-\infty}^{\infty} w(x,y)\; e^{-ikr}/r \; dx\,dy$

where

(5) $r^2 = (x-\xi)^2 + (y-\eta)^2 + \zeta^2$

ξ,η,ζ .=. field points

x,y .=. source points

$k = \omega/c = 2\pi/\lambda$

c .=. velocity of sound in the medium.

Points on the plane are characterized by $\zeta = 0$.

Integral Transforms of the Pressure Distribution

Confining ourselves, for the moment, to the velocity potential ψ_2 on the plane $\zeta = 0$, we have

$$\psi_2(\xi,\eta;k) = \psi(\xi,\eta,0;k).$$

The first step is to take its Fourier Transform

$$(6)\quad \phi_2(\lambda,\mu;k) = \iint_{-\infty}^{\infty} \psi_2(\xi,\eta;k)\, e^{-2\pi i(\lambda\xi+\mu\eta)}\, d\xi\, d\eta .$$

By the Fourier Inversion Theorem we have an integral representation of ψ_2

$$(7)\quad \psi_2(\xi,\eta;k) = \iint_{-\infty}^{\infty} \phi_2(\lambda,\mu;k)\, e^{2\pi i(\lambda\xi+\mu\eta)}\, d\lambda\, d\mu .$$

Applying this idea to equation (4) gives

$$(8)\quad \phi_2(\lambda,\mu;k) = -\tfrac{1}{2\pi}\iiiint_{-\infty}^{\infty} w(x,y)\, e^{-2\pi i(\lambda\xi+\mu\eta)}\, \frac{e^{-ik\sqrt{(x-\xi)^2+(y-\eta)^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\, dx\, dy\, d\xi\, d\eta$$

which reduces to

$$(9)\quad \phi_2(\lambda,\mu;k) = \iint_{-\infty}^{\infty} K_2(\lambda,\mu;x,y;k)\, w(x,y)\, dx\, dy$$

when we define

$$(10)\quad K_2(\lambda,\mu;x,y;k) = -\tfrac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-2\pi i(\lambda\xi+\mu\eta)}\, \frac{e^{-ik\sqrt{(x-\xi)^2+(y-\eta)^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\, d\xi\, d\eta$$

The evaluation of the integral defining K_2 depends on a theorem due to Bochner and is the subject of the next section. For the moment continue with more physical considerations and introduce

$$F_2(\xi,\eta;k) = |\psi_2|^2 = \psi_2(\xi,\eta;k)\, \psi_2^*(\xi,\eta;k)$$

for the square of the magnitude of the velocity potential which is proportional to the square of the magnitude of the pressure. Because our interest is in

limitations in maximum driving power due to the onset of cavitation, it is only the magnitude of p or ψ that is of interest. The introduction of F_2 is the crucial step in the analysis. Continuing with a substitution of the integral representation of ψ_2 we have

$$F_2(\xi,\eta;k) = \iiiint_{-\infty}^{\infty} \phi_2(\lambda,\mu;k)\,\phi_2^*(\lambda',\mu';k)\,e^{2\pi i[(\lambda+\lambda')\xi+(\mu+\mu')\eta]}\,d\lambda\,d\mu\,d\lambda'\,d\mu'.$$

Let

$$\lambda' = \lambda''-\lambda,\quad \mu' = \mu''-\mu\,;\quad \lambda'' = \lambda+\lambda',\quad \mu'' = \mu+\mu'\,;\quad d\lambda' = d\lambda'',\quad d\mu' = d\mu''\,;$$

then

$$F_2(\xi,\eta;k) = \iiiint_{-\infty}^{\infty} \phi_2(\lambda,\mu;k)\,\phi_2^*(\lambda''-\lambda,\mu''-\mu;k)\,e^{2\pi i(\lambda''\xi+\mu''\eta)}\,d\lambda\,d\mu\,d\lambda''\,d\mu''$$

$$= \iint_{-\infty}^{\infty} e^{2\pi i(\lambda''\xi+\mu''\eta)}\,d\lambda''\,d\mu''\cdot\iint_{-\infty}^{\infty} d\lambda\,d\mu\,\phi_2(\lambda,\mu;k)\,\phi_2^*(\lambda''-\lambda,\mu''-\mu;k).$$

We then define

$$\Phi_2(\lambda'',\mu'';k) = \iint_{-\infty}^{\infty} \phi_2(\lambda,\mu;k)\,\phi_2^*(\lambda''-\lambda,\mu''-\mu;k)\,d\lambda\,d\mu$$

and are able to write

$$F_2(\xi,\eta;k) = \iint_{-\infty}^{\infty} \Phi_2(\lambda'',\mu'';k)\,e^{2\pi i(\lambda''\xi+\mu''\eta)}\,d\lambda''\,d\mu''.$$

Finally by substitution

$$\Phi_2(\lambda'',\mu'';k) = \iiiiiint_{-\infty}^{\infty} K_2(\lambda,\mu;x,y;k)\,K_2^*(\lambda''-\lambda,\mu''-\mu;x',y';k)\,w(x,y)\,w^*(x',y')\,dx\,dy\,dx'\,dy'\,d\lambda\,d\mu$$

the Fourier transform of the absolute pressure in terms of the velocity distribution and K_2 function.

A similar analysis will yield the corresponding expressions for $F_3(\xi,\eta,\zeta;k)$ and $\Phi_3(\lambda'',\mu'',\nu'';k)$.

Evaluation of the Transforms by Bochner's Theorem

As before define

$$(11)\quad \phi_3(\lambda,\mu,\nu;k) = \iiint_{-\infty}^{\infty} \psi_3(\xi,\eta,\zeta;k)\,e^{-2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\,d\xi\,d\eta\,d\zeta.$$

Then we have the integral representation

$$(12)\ \psi_3(\xi,\eta,\zeta;k) = \iiint_{-\infty}^{\infty} \phi_3(\lambda,\mu,\nu;k)\, e^{2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\, d\lambda\, d\mu\, d\nu .$$

Again application of this idea to equation (4) gives

$$(13)\ \phi_3(\lambda,\mu,\nu;k) = -\tfrac{1}{2\pi}\iiiint\!\!\int_{-\infty}^{\infty} w(x,y)\, e^{-2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\, \frac{e^{-ik\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}\, dx\, dy\, d\xi\, d\eta\, d\zeta .$$

It then becomes convenient to define

$$(14)\ K_3(\lambda,\mu,\nu;x,y;k) = -\tfrac{1}{2\pi}\iiint_{-\infty}^{\infty} e^{-2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\, \frac{e^{ik\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}\, d\xi\, d\eta\, d\zeta .$$

Both $K_2(\lambda,\mu;x,y;k)$ and $K_3(\lambda,\mu,\nu;x,y;k)$ may be evaluated by a Theorem due to Bochner, e.g., S. Bochner: "Lectures on Fourier Integrals", Annals of Mathematics Studies No. 42, Princeton University Press, 1959; especially § 43, "Trigonometric Integrals in Several Variables". The reference just given is a translation of S. Bochner; "Vorlesungen über Fouriersche Integrale"; Akad. Verlag., Leipzig, 1932, reprinted by Chelsea, N. Y. 1948.

We begin by stating

<u>Bochner's Theorem</u> (Ref. "Fouriersche Integral", pp. 186-187).

If the absolutely integrable function $f(x_1,\dots,x_k)$ depends only on the quantity $r = \sqrt{x_1^2+x_2^2+\cdots+x_k^2}$ i.e.,

$$f(x_1,x_2,\dots,x_k) = \phi\left(\sqrt{x_1^2+\cdots+x_k^2}\right)$$

then the function

$$J(\alpha_1,\dots,\alpha_k) = \int_{-\infty}^{\infty}\!\cdots\!\int f(x_1,\dots,x_k)\, \exp\left(-i\sum_{1}^{k}\alpha_i x_i\right) dx$$

in a similar manner depends only on the quantity,

$$\alpha = \sqrt{\alpha_1^2+\cdots+\alpha_k^2}$$

Furthermore, one can replace the original k-fold integral for $J(\alpha)$ by the

simpler expression

$$J(\alpha) = (2\pi)^{k/2} / \alpha^{\frac{k-2}{2}} \int_0^\infty \phi(\rho)\rho^{k/2} J_{\frac{k-2}{2}}(\alpha\rho)\, d\rho$$

where $J_\mu(t)$ is the Bessel function of the μ^{th} order. If one further introduces the quantity $s = \alpha^2 = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_k^2$ one discovers that

1.) For $k = 2m+2$ $\quad J(\alpha) = (-1)^m \pi^{m+1} 2^{2m+1} \, d^m/ds^m \int_0^\infty \phi(\rho)\rho \, J_0(\sqrt{s}\rho)\, d\rho.$

2.) For $k = 2m+1$ $\quad J(\alpha) = (-1)^m \pi^m 2^{2m+1} \, d^m/ds^m \int_0^\infty \phi(\rho) \cos(\sqrt{s}\rho)\, d\rho.$

From eq. (14) we have

$$K_3(\lambda,\mu,\nu;\, x,y;\, k) = -\,{}^1\!/_{2\pi} \iiint_{-\infty}^{\infty} e^{-2\pi i(\lambda\xi+\mu\eta+\nu\zeta)} \frac{e^{-ik\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}}{\sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2}}\, d\xi\, d\eta\, d\zeta.$$

To apply Bochner's Theorem to this we make the following substitutions:

$2\pi\lambda = \alpha_1 \qquad \xi = x + x_1$

$2\pi\mu = \alpha_2 \qquad \eta = y + x_2$

$2\pi\nu = \alpha_3 \qquad \zeta = x_3$

$$(15)\; K_3(\lambda,\mu,\nu;\, x,y;\, k) = -\,{}^1\!/_{2\pi}\, e^{-i(\alpha_1 x+\alpha_2 y)} \iiint_{-\infty}^{\infty} e^{-i(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3)} \frac{e^{-ik\sqrt{x_1^2+x_2^2+x_3^2}}}{\sqrt{x_1^2+x_2^2+x_3^2}}\, dx_1\, dx_2\, dx_3.$$

By Bochner's Theorem,

$$f(x_1, x_2, \ldots, x_k) = \phi\left(\sqrt{x_1^2+\cdots+x_k^2}\right) = \phi(r) = e^{-ikr}/r$$

where

$$r = \sqrt{x_1^2+x_2^2+x_3^2}\,.$$

$k = 3 = 2m+1 \quad \therefore \quad m = 1$

$$J(\alpha) = -8\pi\, d/ds_3 \int_0^\infty e^{-ik\rho}/\rho \, \cos(\sqrt{s_3}\rho)\, d\rho.$$

Our expression becomes

$$(16)\; K_3(\lambda,\mu,\nu;\, x,y;\, k) = 4\, e^{-2\pi i(\lambda x+\mu y)}\, d/ds_3 \int_0^\infty e^{-ik\rho}/\rho \, \cos(\sqrt{s_3}\rho)\, d\rho$$

where $s_3 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (a\pi)^2(\lambda^2+\mu^2+\nu^2)$.

The problem that remains is to evaluate

$$d/ds_3 \int_0^\infty e^{-ik\rho}/\rho \quad \cos(\sqrt{s_3}\,\rho)\,d\rho.$$

Let

$$I(k, s_3) = d/ds_3 \int_0^\infty e^{-ik\rho}/\rho \quad \cos(\sqrt{s_3}\,\rho)\,d\rho.$$

The integral expression may be written in the following form:

$$\int_0^\infty \frac{e^{-ik\rho}}{\rho} \cos(\sqrt{s_3}\,\rho)\,d\rho = -\int_0^\infty \cos(k\rho)[1-\cos\sqrt{s_3}\,\rho]/\rho \; d\rho \; +$$

$$+ \int_0^\infty \cos(k\rho)/\rho \; d\rho \; - i\int_0^\infty \sin(k\rho)/\rho \quad \cos(\sqrt{s_3}\,\rho)\,d\rho.$$

From Oberhettinger, Fritz: "Tabellen zur Fourier Transformation", Springer-Verlag, Berlin 1957, p. 20

$$\int_0^\infty \cos(k\rho)[1-\cos\sqrt{s_3}\,\rho]/\rho \; d\rho = 1/2 \ln|1 - s_3/k^2|.$$

Thus

$$I(k,s_3) = d/ds_3\left[-1/2\ln|1-s_3/k^2|\right] + \left[\infty;\ \begin{matrix}\text{a constant}\\ \text{w.r.t } d/ds_3\end{matrix}\right] - i\, d/ds_3 \int_0^\infty \frac{\sin(k\rho)}{\rho}\cos(\sqrt{s_3}\,\rho)\,d\rho.$$

$$d/ds_3 \int_0^\infty \sin(k\rho)/\rho \quad \cos(\sqrt{s_3}\,\rho)\,d\rho = -\int_0^\infty \frac{\sin(k\rho)}{\rho} \; \sin(\sqrt{s_3}\,\rho) \; \rho/2\sqrt{s_3} \; d\rho =$$

$$= -1/2\sqrt{s_3} \int_0^\infty \sin(k\rho)\sin(\sqrt{s_3}\,\rho)\,d\rho.$$

From the Vorwort of the preceding reference, the function

$$g(y) = \int_0^\infty f(x)\sin(xy)\,dx$$

after a Fourier-Sine Transformation is

$$f(x) = 2/\pi \int_0^\infty g(y)\sin(xy)\,dy.$$

Then
$$f(x) = 2/\pi \int_0^\infty \sin(xy)\,dy \int_0^\infty f(x')\sin(x'y)\,dx'$$
$$= \int_0^\infty f(x')\,dx' \; 2/\pi \int_0^\infty \sin(xy)\sin(x'y)\,dy$$
$$= \int_0^\infty \delta_s(x-x')\,f(x')\,dx' \;; \quad x \geq 0.$$

By definition
$$2/\pi \int_0^\infty \sin(xy)\sin(x'y)\,dy = \delta_s(x-x').$$

Applying this to our integral
$$-1/2\sqrt{s_3}\int_0^\infty \sin(k\rho)\sin(\sqrt{s_3}\,\rho)\,d\rho = -\pi/4\sqrt{s_3}\quad \delta_s(k-\sqrt{s_3}).$$

Therefore
$$I(k, s_3) = \frac{1}{2(k^2-s_3)} + \frac{\pi i}{4\sqrt{s_3}}\;\delta_s(k-\sqrt{s_3}).$$

Equation (16) becomes

$$(17)\; K_3(\lambda,\mu,\nu;x,y;k) = e^{-2\pi i(\lambda x+\mu y)}\left[\frac{2}{k^2-s_3} + \frac{\pi i}{\sqrt{s_3}}\;\delta_s(k-\sqrt{s_3})\right]$$

where $s_3 = (2\pi)^2(\lambda^2+\mu^2+\nu^2)$.

Returning now to equation (10) and applying Bochner's theorem using the same substitutions as before

$$(18)\; K_2(\lambda,\mu;x,y;k) = -e^{-2\pi i(\lambda x+\mu y)}\int_0^\infty e^{-ik\rho}\,J_0(\sqrt{s_2}\,\rho)\,d\rho.$$

From Oberhettinger, Fritz; "Tabellen zur Fourier Transformation", Springer-Verlag, Berlin, 1957, p. 61

$$\int_0^\infty J_0(\sqrt{s_2}\,\rho)\cos(k\rho)\,d\rho = \begin{cases}(s_2-k^2)^{-\frac{1}{2}} & 0 < k < \sqrt{s_2}\\ 0 & k > \sqrt{s_2}\end{cases}$$

Also from p. 165

$$\int_0^\infty J_0(\sqrt{s_2}\,\rho)\sin(k\rho)\,d\rho = \begin{cases}0 & 0 < k < \sqrt{s_2}\\ (k^2-s_2)^{-\frac{1}{2}} & k > \sqrt{s_2}\end{cases}$$

Therefore

$$(19)\; K_2(\lambda,\mu;x,y;k) = -\frac{1}{\sqrt{s_2-k^2}}\;e^{-2\pi i(\lambda x+\mu y)} \qquad k^2 \neq s_2$$

where $s_2 = (2\pi)^2(\lambda^2+\mu^2)$.

Compatability Check between K_2 and K_3

K_2 and K_3 are related and this relation furnishes a valuable check on some rather tricky integration techniques used to obtain them. The relation between them is

$$K_2(\lambda,\mu;\, x,y;\, k) = \int_{-\infty}^{\infty} K_3(\lambda,\mu,\nu;\, x,y;\, k)\, d\nu.$$

To prove this, begin with

$$K_2(\lambda,\mu;\, x,y;\, k) = -\frac{1}{\sqrt{s_2 - k^2}}\, e^{-2\pi i(\lambda x+\mu y)}$$

$$K_3(\lambda,\mu,\nu;\, x,y;\, k) = \left[\frac{2}{k^2 - s_3} + \frac{\pi i}{\sqrt{s_3}}\,\delta_s(k - \sqrt{s_3})\right] e^{-2\pi i(\lambda x+\mu y)}.$$

Since the exp $[-2\pi i(\lambda x+\mu y)]$ factor does not contain ν, let

$$L_2 = -\frac{1}{\sqrt{s_2 - k^2}}, \quad s_2 = (2\pi)^2(\lambda^2+\mu^2)$$

$$L_3 = \left[\frac{2}{k^2 - s_3} + \frac{\pi i}{\sqrt{s_3}}\,\delta_s(k - \sqrt{s_3})\right], \quad s_3 = (2\pi)^2(\lambda^2+\mu^2+\nu^2).$$

Therefore the relation that must be shown is

$$L_2 = \int_{-\infty}^{\infty} L_3\, d\nu.$$

The right side of the equation is equal to

$$\int_{-\infty}^{\infty} \frac{2}{[k^2-(2\pi)^2(\lambda^2+\mu^2+\nu^2)]}\, d\nu + \int_{-\infty}^{\infty} \frac{\pi i}{2\pi\sqrt{\lambda^2+\mu^2+\nu^2}}\,\delta_s(k - 2\pi\sqrt{\lambda^2+\mu^2+\nu^2})\, d\nu.$$

An analysis of the first integral yields

$$I_1 = \int_{-\infty}^{\infty} \frac{2}{[k^2-(2\pi)^2(\lambda^2+\mu^2+\nu^2)]}\, d\nu = \int_{-\infty}^{\infty} \frac{2}{[k^2-(2\pi)^2(\lambda^2+\mu^2)]-(2\pi)^2\nu^2}\, d\nu$$

$$= \int_{-\infty}^{\infty} \frac{2}{a^2-(2\pi\nu)^2}\, d\nu \quad \text{where} \quad a^2 = k^2 - s_2.$$

Let $x = 2\pi\nu$, $dx = 2\pi\, d\nu$. Substitution gives

$$1/\pi \int_{-\infty}^{\infty} \frac{dx}{a^2 - x^2} = 2/\pi \int_0^{\infty} \frac{dx}{a^2 - x^2} .$$

Let $x/a = \xi$, $\frac{dx}{a} = d\xi$ Therefore

$$2/\pi a \int_0^{\infty} \frac{d\xi}{1-\xi^2} = 1/\pi a \Big|_0^{\infty} \ln \frac{1+\xi}{1-\xi} .$$

From Ryshik, I. M. and Gradstein, I. S., "Summen; Produkt und Integral Tafeln", Veb. Deutscher Verlag der Wissenschaften, Berlin, 1957, No. 1.513.2

$$\lim_{\xi \to \infty} \ln \frac{1+\xi}{1-\xi} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)\xi^{2k-1}} = 0.$$

Therefore the first integral has the value zero.

Since the second integral involves an even function

$$I_2 = \int_{-\infty}^{\infty} \frac{\pi i}{2\pi\sqrt{\lambda^2+\mu^2+\nu^2}} \delta_s\left(k - 2\pi\sqrt{\lambda^2+\mu^2+\nu^2}\right) d\nu = 2\int_0^{\infty} \frac{\pi i}{2\pi\sqrt{\lambda^2+\mu^2+\nu^2}} \delta_s\left(k - 2\pi\sqrt{\lambda^2+\mu^2+\nu^2}\right) d\nu.$$

With the following substitutions:

$$x = 2\pi\sqrt{\lambda^2+\mu^2+\nu^2}, \quad x^2 = (2\pi)^2(\lambda^2+\mu^2+\nu^2), \quad \nu = \frac{\sqrt{x^2-(2\pi)^2(\lambda^2+\mu^2)}}{2\pi}$$

$$dx = \frac{(2\pi)^2 \nu\, d\nu}{2\pi\sqrt{\lambda^2+\mu^2+\nu^2}}, \quad x\,dx = (2\pi)^2 \nu\, d\nu, \quad d\nu = x\, dx/(2\pi)^2\nu$$

$$I_2 = i\int_{\sqrt{S_2}}^{\infty} \delta_s(k-x) \frac{dx}{\sqrt{x^2 - (2\pi)^2(\lambda^2+\mu^2)}} .$$

In the previous section the function δ_s was defined as

$$\delta_s(k-x) = 2/\pi \int_0^{\infty} \sin(ky)\sin(xy)\, dy .$$

We shall show later in the Appendix that I_2 can be evaluated by the direct substitution of this quantity into the integral. Here we shall treat δ_s as the Dirac delta function and because of the limits involved

$$I_2 = \begin{cases} 0 & k < \sqrt{S_2} \\ \dfrac{i}{\sqrt{k^2 - S_2}} & k > \sqrt{S_2} \end{cases} .$$

The integral $\int_{-\infty}^{\infty} L_3 \, d\nu$, then, has the value $\frac{i}{\sqrt{k^2 - s_2}}$ or $-\frac{1}{\sqrt{s_2 - k^2}}$ when $k > \sqrt{s_2}$ and thus is equal to L_2.

Comparison with King's Results for a Circular Piston

Louis V. King wrote an excellent paper "On the Acoustic Radiation Field of the Piezo-Electric Oscillator and the Effect of Viscosity on Transmission", Canadian Jour. of Research II, 135-155 (1934). He obtains the expression:

$$\phi = a\dot{x}\int_0^{\infty} e^{-\mu z} J_0(\lambda\rho)\, J_1(\lambda a)\, \frac{d\lambda}{\mu}$$

for the velocity potential due to the disc of radius "a" oscillating with simple harmonic motion and maximum velocity $\dot{x}$. These symbols need translating for comparison with our work and this will be done after obtaining the same result by our method.

For the King case

$$w(x,y) = \begin{cases} W_0 & \text{for } x^2+y^2 \le a^2 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\tilde{\phi}_3(\lambda,\mu,\nu;k) = \iint_{-\infty}^{\infty} w(x,y)\, K_3(\lambda,\mu,\nu, x,y;k)\, dx\, dy .$$

But

$$K_3(\lambda,\mu,\nu; x,y; k) = e^{-2\pi i(\lambda x + \mu y)} L_3$$

where $L_3 = \left[\frac{2}{k^2 - s_3} + \frac{\pi i}{\sqrt{s_3}}\, \delta_3(k - \sqrt{s_3})\right]$.

Since L_3 does not contain x and y then by substitution

$$\phi_3(\lambda,\mu,\nu;k) = W_0 L_3 \iint_{[x,y;\, x^2+y^2 < a^2]^*} e^{-2\pi i(\lambda x + \mu y)}\, dx\, dy .$$

* Read [P; condition on P] .=. "the set of points P such that the condition on P is satisfied".

This expression has the correct symmetry for an application of Bochner's Theorem where

$$\phi\left(\sqrt{x^2+y^2}\right) = \begin{cases} 1 & \text{when } x^2+y^2 < a^2 \\ 0 & \text{otherwise} \end{cases}.$$

With the proper substitutions and application of the theorem

$$\phi_3(\lambda,\mu,\nu;k) = W_0 L_3\left[2\pi \int_0^a \rho J_0(\sqrt{s_2}\,\rho)\,d\rho\right].$$

The expression

$$2\pi \int_0^a \rho J_0(\sqrt{s_2}\,\rho)\,d\rho = 2\pi/(\sqrt{s_2})^2 \int_0^a (\sqrt{s_2}\,\rho) J_0(\sqrt{s_2}\,\rho)\sqrt{s_2}\,d\rho.$$

which is evaluated by No. 835.1 from "Tables of Integrals and Other Mathematical Data", Dwight, Herbert Bristol, MacMillan Co. New York to be

$$2\pi/(\sqrt{s_2})^2 \left|_0^a \sqrt{s_2}\,\rho\, J_1(\sqrt{s_2}\,\rho) = 2\pi a/\sqrt{s_2}\;\; J_1(\sqrt{s_2}\,a).\right.$$

Thus

$$\phi_3(\lambda,\mu,\nu;k) = W_0 L_3 \frac{2\pi a}{\sqrt{s_2}} J_1(\sqrt{s_2}\,a).$$

From Equation (12)

$$\psi_3(\xi,\eta,\zeta;k) = \iiint_{-\infty}^{\infty} \phi_3(\lambda,\mu,\nu;k) e^{2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\,d\lambda\,d\mu\,d\nu.$$

Substitution yields

$$\psi_3(\xi,\eta,\zeta;k) = W_0 \iiint_{-\infty}^{\infty} \frac{2\pi a}{\sqrt{s_2}} J_1(\sqrt{s_2}\,a)\left[\frac{2}{k^2-s_3} + \frac{\pi i}{\sqrt{s_3}}\delta_s(k-\sqrt{s_3})\right] e^{2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\,d\lambda\,d\mu\,d\nu$$

where $s_2 = (2\pi)^2(\lambda^2+\mu^2)$

$$s_3 = (2\pi)^2(\lambda^2+\mu^2+\nu^2).$$

The first step in the evaluation of $\psi_3(\xi,\eta,\zeta;k)$ is to consider the integral

$$\int_{-\infty}^{\infty} \frac{2}{k^2-s_3} + \frac{\pi i}{\sqrt{s_3}}\delta_s(k-\sqrt{s_3})e^{2\pi i\zeta\nu}\,d\nu.$$

Since $S_3 = (2\pi)^2(\lambda^2+\mu^2) + (2\pi)^2\nu^2 = S_2 + (2\pi)^2\nu^2$ we have

$$\int_{-\infty}^{\infty} \frac{2}{k^2 - S_2 - (2\pi)^2\nu^2}\, e^{2\pi i \xi \nu}\, d\nu + \int_{-\infty}^{\infty} \frac{\pi i}{\sqrt{S_2 + (2\pi)^2\nu^2}}\, \delta_s\left(k - \sqrt{S_2 + (2\pi)^2\nu^2}\right) e^{2\pi i \xi \nu}\, d\nu$$

where we will denote the first integral by A and the second by B.

$$A = \int_{-\infty}^{\infty} \frac{2}{k^2 - S_2 - (2\pi)^2\nu^2}\, e^{2\pi i \nu \xi}\, d\nu = 2\int_0^{\infty} \frac{2}{k^2 - S_2 - (2\pi)^2\nu^2} \cos(2\pi\nu\xi)\, d\nu.$$

Let $x = 2\pi\nu$

$x^2 = (2\pi)^2\nu^2$

$dx = 2\pi\, dx$

$a^2 = k^2 - S_2$.

Then

$$A = 2/\pi \int_0^{\infty} \frac{1}{a^2 - x^2} \cos(x\xi)\, dx.$$

From "Tabellen zur Fourier Transformation", p.3

$$\int_0^{\infty} (a^2 - x^2)^{-1} \cos(xy)\, dx = \pi/2a \;\; \sin(ay).$$

Then $A = \dfrac{\sin \xi\sqrt{k^2 - S_2}}{\sqrt{k^2 - S_2}}$ when $\xi > 0$.

$$B = \int_{-\infty}^{\infty} \frac{\pi i}{\sqrt{S_2 + (2\pi)^2\nu^2}}\, \delta_s\left(k - \sqrt{S_2 + (2\pi)^2\nu^2}\right) e^{2\pi i \xi \nu}\, d\nu =$$

$$= 2\int_0^{\infty} \frac{\pi i}{\sqrt{S_2 + (2\pi)^2\nu^2}}\, \delta_s\left(k - \sqrt{S_2 + (2\pi)^2\nu^2}\right) \cos(2\pi\xi\nu)\, d\nu.$$

Let

$X = \sqrt{S_2 + (2\pi)^2\nu^2}$

$dx = \dfrac{(2\pi)^2\nu\, d\nu}{\sqrt{S_2 + (2\pi)^2\nu^2}}$

$x\, dx = (2\pi)^2\nu\, d\nu$

$\nu = \dfrac{\sqrt{x^2 - S_2}}{2\pi}$

Then

$$B = i\int_{\sqrt{s_2}}^{\infty} \frac{\cos \zeta\sqrt{x^2 - s_2}}{\sqrt{x^2 - s_2}}\, \delta_s(k - x)\, dx .$$

Treating $\delta_s(k - x)$ as we did in the previous section

$$B = i \begin{cases} \dfrac{\cos \zeta\sqrt{k^2 - s_2}}{\sqrt{k^2 - s_2}} & k > \sqrt{s_2} \\ 0 & k < \sqrt{s_2} \end{cases}$$

The integral under consideration then is

$$A + B = \frac{\sin \zeta\sqrt{k^2 - s_2}}{\sqrt{k^2 - s_2}} + i\,\frac{\cos \zeta\sqrt{k^2 - s_2}}{\sqrt{k^2 - s_2}}\,, \qquad k > \sqrt{s_2}$$

$$A + B = \frac{i e^{-i\zeta\sqrt{k^2 - s_2}}}{\sqrt{k^2 - s_2}}\,, \qquad k > \sqrt{s_2} .$$

Therefore

$$\psi_s(\xi, \eta, \zeta; k) = W_0 \iint_{-\infty}^{\infty} \frac{2\pi a}{\sqrt{s_2}}\, J_1(\sqrt{s_2}\, a) \cdot \frac{i e^{-i\zeta\sqrt{k^2 - s_2}}}{\sqrt{k^2 - s_2}} \cdot e^{2\pi i(\lambda\xi + \mu\eta)}\, d\lambda\, d\mu .$$

Again $s_2 = (2\pi)^2(\lambda^2 + \mu^2)$ and we have

$$\psi_s(\xi, \eta, \zeta; k) = W_0 \iint_{-\infty}^{\infty} \frac{a}{\sqrt{\lambda^2 + \mu^2}}\, J_1(2\pi a\sqrt{\lambda^2 + \mu^2}\,)\, \frac{e^{i\zeta\sqrt{s_2 - k^2}}}{\sqrt{s_2 - k^2}}\, e^{2\pi i(\lambda\xi + \mu\eta)}\, d\lambda\, d\mu .$$

which is suitable for the application of Bochner's Theorem.

Let

$$x_1 = 2\pi\lambda \qquad d\lambda = \frac{dx_1}{2\pi} \qquad \xi = \alpha_1 \qquad \alpha = \sqrt{\alpha_1^2 + \alpha_2^2} = \sqrt{\xi^2 + \eta^2}$$

$$x_2 = 2\pi\mu \qquad d\mu = \frac{dx_2}{2\pi} \qquad \eta = \alpha_2 \qquad r = \sqrt{x_1^2 + x_2^2} = 2\pi\sqrt{\lambda^2 + \mu^2} = \sqrt{s_2}$$

and we obtain by substitution

$$\psi_s(\xi, \eta, \zeta; k) = \frac{a W_0}{2\pi} \iint_{-\infty}^{\infty} {}^1/r \;\; J_1(ra)\, \frac{e^{-\zeta\sqrt{r^2 - k^2}}}{\sqrt{r^2 - k^2}}\; e^{i(\alpha_1 x_1 + \alpha_2 x_2)}\, dx_1\, dx_2 .$$

Application of Bochner's Theorem yields

$$\psi_s(\xi, \eta, \zeta; k) = a W_0 \int_0^{\infty} \frac{J_1(ar)\, e^{-\zeta\sqrt{r^2 - k^2}}}{\sqrt{r^2 - k^2}}\, J_0(\sqrt{\xi^2 + \eta^2}\; r)\, dr .$$

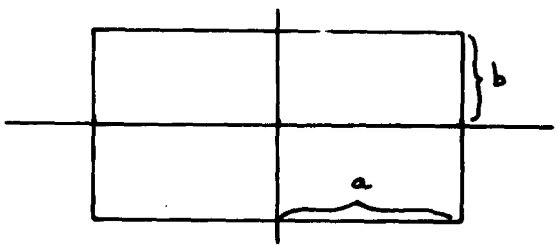
If we let $\rho = \sqrt{\xi^2 + \eta^2}$, $\mu = \sqrt{r^2 - k^2}$

$$\psi_s(\xi, \eta, \zeta; k) = a\omega_0 \int_0^\infty \frac{e^{-\mu\zeta}}{\mu} J_1(ra)\, J_0(r\rho)\, dr$$

which is comparable to King's results.

Part II

The next case to be considered is the investigation of the velocity potential for a rectangular piston, centered at the origin with half-sides "a", "b" as shown in the figure.



We start again with the equations derived in the previous section:

$$(1)\quad \Phi_s(\lambda, \mu, \nu; k) = \iint_{-\infty}^{\infty} w(x,y)\, K_s(\lambda, \mu, \nu; x, y; k)\, dx\, dy$$

$$(2)\quad K_s(\lambda, \mu, \nu; x, y; k) = e^{-2\pi i(\lambda x + \mu y)} L_s$$

$$(3)\quad L_s = \left[\frac{2}{k^2 - s_s} + \frac{\pi i}{\sqrt{s_s}}\, \delta_s(k - \sqrt{s_s})\right].$$

For he pressure boundary condition we take

$$w(x, y) = \begin{cases} 1 & \text{for points } (x, y) \text{ inside the rectangle} \\ 0 & \text{for points } (x, y) \text{ outside the rectangle.} \end{cases}$$

Substitution in equation (1) yields

$$(4)\quad \Phi_s(\lambda, \mu, \nu; k) = \left[\frac{2}{k^2 - s_s} + \frac{\pi i}{\sqrt{s_s}}\, \delta_s(k - \sqrt{s_s})\right] \int_{-a}^{a}\int_{-b}^{b} e^{-2\pi i(\lambda x + \mu y)}\, dx\, dy.$$

After integration the expression becomes

$$(5)\quad \phi_s(\lambda,\mu,\nu;k) = \left[\frac{2}{k^2 - s_s} + \frac{\pi i}{\sqrt{s_s}}\,\delta_s(k-\sqrt{s_s})\right]\frac{\sin(2\pi\mu b)\,\sin(2\pi\lambda a)}{\pi^2\mu\lambda}.$$

The general integral representation for the velocity potential was found to be

$$\psi_s(\xi,\eta,\zeta;k) = \iiint_{-\infty}^{\infty}\phi_s(\lambda,\mu,\nu;k)\,e^{2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\,d\lambda\,d\mu\,d\nu.$$

Substitution of equation (5) into this expression gives us the velocity potential for the rectangle

$$(6)\quad \psi_s(\xi,\eta,\zeta;k) = \iiint_{-\infty}^{\infty}\frac{\sin(2\pi\mu b)\,\sin(2\pi\lambda a)}{\pi^2\mu\lambda}\,e^{2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\cdot$$

$$\cdot\left[\frac{2}{k^2 - s_s} + \frac{\pi i}{\sqrt{s_s}}\,\delta_s(k-\sqrt{s_s})\right]d\lambda\,d\mu\,d\nu.$$

or

$$(7)\quad \psi_s(\xi,\eta,\zeta;k) = \iint_{-\infty}^{\infty}\frac{\sin(2\pi\mu b)\,\sin(2\pi\lambda a)}{\pi^2\mu\lambda}\,e^{2\pi i(\lambda\xi+\mu\eta)}\,d\lambda\,d\mu\,\cdot$$

$$\cdot\int_{-\infty}^{\infty}\left[\frac{2}{k^2 - s_s} + \frac{\pi i}{\sqrt{s_s}}\,\delta_s(k-\sqrt{s_s})\right]e^{2\pi i\nu\zeta}\,d\nu.$$

The evaluation of the second integral was carried out in detail in Part I and found to be equal to the following expression when $k > 2\pi\sqrt{\lambda^2+\mu^2}$:

$$\frac{i e^{-i\zeta\sqrt{k^2-(2\pi)^2(\lambda^2+\mu^2)}}}{\sqrt{k^2-(2\pi)^2(\lambda^2+\mu^2)}}.$$

After substitution and separation of variables our expression is of the form

$$(8)\ \psi_3(\xi,\eta,\zeta;k)=\int_{-\infty}^{\infty}\frac{\sin(2\pi a\lambda)}{\pi^2\lambda}e^{2\pi i\lambda\xi}\,d\lambda\int_{-\infty}^{\infty}\frac{i\sin(2\pi b\mu)}{\mu\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}e^{i\left(2\pi\eta\mu-\zeta\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}\right)}\,d\mu.$$

Let

$$I=\int_{-\infty}^{\infty}\frac{i\sin(2\pi b\mu)}{\mu\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}e^{i\left(2\pi\eta\mu-\zeta\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}\right)}d\mu$$

Then

$$I=\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-2\pi ib\mu}\cdot e^{i\left(2\pi\eta\mu-\zeta\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}\right)}}{\mu\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}\,d\mu\ -$$

$$-\ \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-2\pi ib\mu}\cdot e^{i\left(2\pi\eta\mu-\zeta\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}\right)}}{\mu\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}\,d\mu.$$

Let

$$A=\int_{-\infty}^{\infty}\frac{e^{2\pi i\mu(b+\eta)}\cdot e^{-i\zeta\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}}{\mu\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}\,d\mu.$$

Introduce

$$b+\eta=c\ ,\qquad a^2=k^2-(2\pi\lambda)^2.$$

Then

$$A=\int_{-\infty}^{\infty}\frac{e^{-i\zeta\sqrt{a^2-(2\pi\mu)^2}}}{\mu\sqrt{a^2-(2\pi\mu)^2}}e^{2\pi i\mu c}\,d\mu.$$

Taking the derivative of **A** with respect to c yields

$$d/dc\,(A)=2\pi i\int_{-\infty}^{\infty}\frac{e^{-i\zeta\sqrt{a^2-(2\pi\mu)^2}}}{\sqrt{a^2-(2\pi\mu)^2}}e^{2\pi i\mu c}\,d\mu$$

or

$$d/_{dc}\,(A) = 2\pi i\int_{-\infty}^{\infty}\frac{e^{\xi\sqrt{(ia)^2-(2\pi i\mu)^2}}}{i\sqrt{(ia)^2-(2\pi i\mu)^2}}\,e^{2\pi i\mu c}\,d\mu .$$

From Campbell and Foster, <u>Fourier Integrals for Practical Applications</u>, 1942 printing, p. 111, pair No. 868

$$F(f) = \frac{e^{-\sigma(\rho^2-p^2)^{\frac{1}{2}}}}{(\rho^2-p^2)^{1/2}} \qquad\qquad G(g) = 1/\pi\;\; K_0\left[\rho(g^2+\sigma^2)^{\frac{1}{2}}\right]$$

where $p = 2\pi i f$; ρ and σ are complex numbers, not infinite and the real parts are not less than 0.

Applying this to our expression $\sigma = -\xi$, $\rho = ia$

$$d/_{dc}\,(A) = 2K_0\left[i\sqrt{k^2-(2\pi\lambda)^2}\,\sqrt{c^2+\xi^2}\right].$$

Therefore

$$A = 2\int_{-\infty}^{c=b+\eta} K_0\left[i\sqrt{k^2-(2\pi\lambda)^2}\,\sqrt{c^2+\xi^2}\right]dc .$$

Similarly let

$$B = \int_{-\infty}^{\infty}\frac{e^{2\pi i\mu(\eta-b)}\cdot e^{-i\xi\sqrt{\{k^2-(2\pi\lambda)^2\}-(2\pi\mu)^2}}}{\mu\sqrt{a^2-(2\pi\mu)^2}}\,d\mu .$$

Introduce

$$c = \eta - b, \qquad a^2 = k^2-(2\pi\lambda)^2 .$$

Then

$$B = \int_{-\infty}^{\infty}\frac{e^{2\pi i\mu c}\cdot e^{-i\xi\sqrt{a^2-(2\pi\mu)^2}}}{\mu\sqrt{a^2-(2\pi\mu)^2}}\,d\mu .$$

Applying the same method as before

$$B = 2\int_{-\infty}^{c=\eta-b} K_0\left[i\sqrt{k^2-(2\pi\lambda)^2}\,\sqrt{c^2+\xi^2}\right]dc .$$

The expression $I = 1/2\,A - 1/2\,B$ then, becomes

$$I = \int_{-\infty}^{\eta+b} K_0\left[i\sqrt{k^2-(2\pi\lambda)^2}\,\sqrt{c^2+\xi^2}\right]dc - \int_{-\infty}^{\eta-b} K_0\left[i\sqrt{k^2-(2\pi\lambda)^2}\,\sqrt{c^2+\xi^2}\right]dc .$$

Therefore

$$I = \int_{\eta-b}^{\eta+b} K_0\left[i\sqrt{k^2-(2\pi\lambda)^2}\,\sqrt{c^2+\xi^2}\right]dc .$$

Substitution into equation (8) yields

$$(9)\ \psi_3(\xi,\eta,\zeta;k) = \int_{-\infty}^{\infty} \frac{\sin(a\pi a\lambda)}{\pi^2\lambda} e^{a\pi i\lambda\xi} \int_{\eta-b}^{\eta+b} K_0\left[i\sqrt{k^2-(a\pi\lambda)^2}\sqrt{c^2+\zeta^2}\right] dc .$$

Our final expression, then, for the velocity potential of the rectangle is given by

$$(10)\ \psi_3(\xi,\eta,\zeta;k) = \int_{-\infty}^{\infty}\int_{\eta-b}^{\eta+b} \frac{\sin(a\pi a\lambda)}{\pi^2\lambda} \cos a\pi\lambda\xi \cdot K_0\left[i\sqrt{k^2-(a\pi\lambda)^2}\sqrt{c^2+\zeta^2}\right] dc\, d\lambda .$$

This is only a tentative solution to the problem of the rectangle which is analogous to King's solution for the circle. Further study is necessary in order to determine whether this double integral can be reduced to a simpler form.

Part III

In this section we discuss briefly a certain alternative formulation of the problem. Work on this has not yet progressed to the point where it is possible to make a definite statement as to its value, but we feel that it merits further consideration.

Let D be a domain in the x,y plane. The basic expression for the velocity potential for the domain D is, from Part I,

$$(1)\quad \psi(\xi,\eta,\zeta;k) = -1/2\pi \iint_{-\infty}^{\infty} w(x,y)\, e^{-ikr}/r \; dx\, dy$$

in which $w(x,y)$ is the boundary pressure distribution normal to D, and

$$r = \sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2} .$$

By a change of variables, we may take $\xi = 0$, $\eta = 0$ so that (1) assumes the form:

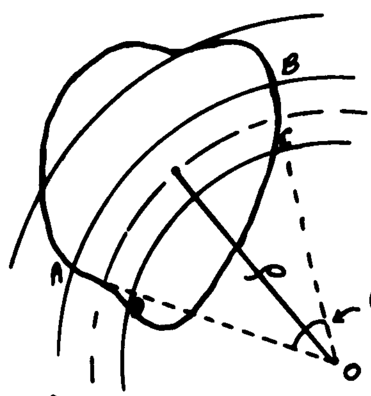
$$(2)\quad \psi(\zeta;k) = -1/2\pi \iint_{-\infty}^{\infty} w(x,y)\, e^{-ikr}/r \; dx\, dy$$

in which $r = \sqrt{x^2+y^2+\zeta^2}$. Taking, as before, $w(x,y) = 1$ if the point (x,y) is in the domain D and $w(x,y) = 0$ otherwise, we have

$$(3)\quad \psi(\zeta;k) = -\,1/2\pi \iint_D \frac{e^{-ikr}}{r}\,dx\,dy$$

where now the double integration is extended over the domain D.

Let $\rho = \sqrt{x^2+y^2}$. Then the function e^{-ikr}/r is constant on circles of radius ρ centered at the origin. This means that the integral (3) may be evaluated by summing up the contributions to it from annular bands, centered at the origin, in which e^{-ikr}/r is approximately constant. Thus, referring to the figure, we have that the contribution due to the annular band ABCD is:



$$\psi_{ABCD} = -\,1/2\pi\,\frac{e^{-ikr}}{r}\,\theta\cdot\rho\,d\rho$$

where θ is the central angle subtended by the domain D at the distance from the origin. We define the function $f_D(\rho)$ to be that fraction of the circumference of the circle of radius ρ, centered at the origin, which is intercepted by the domain. It is easy to see that $f_D(\rho) = \theta/2\pi$, so that the expression for the contribution due to ABCD becomes

$$\psi_{ABCD} = -\,\frac{e^{-ikr}}{r}\,f_D(\rho)\,\rho\,d\rho.$$

The values $f_D(\rho)$ are zero for those ρ such that the circle of radius ρ does not meet the domain D . Summing up the contributions, we have, using the fact that $\rho\,d\rho = r\,dr$

$$\psi(\zeta;k) = -\int_0^\infty f_D(\rho)\,e^{-ikr}\,dr$$

and then, since $\rho = \sqrt{r^2 - \zeta^2}$

$$(4) \quad \psi(\zeta;k) = -\int_{\zeta}^{\infty} f_D\left(\sqrt{r^2-\zeta^2}\right) e^{-ikr}\, dr.$$

This is the basic expression in our new formulation. We see that the problem of evaluating the velocity potential ψ for a given domain D splits into two parts:

I) find the function $f_D(\rho)$ for the given plane domain D. This is the purely geometric part of the problem;

II) carry out the integration in (4). This is the analytic side of the problem, and the more difficult. Without going into too much detail, we give here a few brief remarks concerning each of these two parts.

I). If D is a disc of radius a, centered at the origin, it is easy to see that $f_D(\rho) = 1$ for $0 \leq \rho \leq a$, and $f_D(\rho) = 0$ otherwise. In this case, the integral (4) becomes:

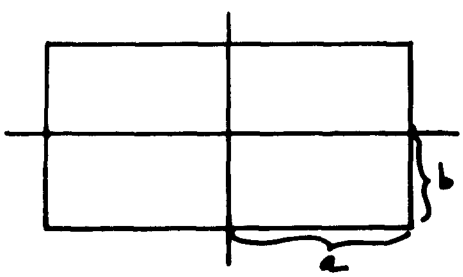
$$\psi(\zeta;k) = -\int_{\zeta}^{\sqrt{\zeta^2+a^2}} e^{-ikr}\, dr$$

and this may be explicitly evaluated to yield:

$$\psi(\zeta;k) = 1/ik \left\{ e^{-ik\sqrt{\zeta^2+a^2}} - e^{-ik\zeta} \right\}.$$

This result agrees with that of Stenzel-Brosze, Leitfaden zur Berechnung von Schallvorgängen, Springer-Verlag, 1958; Part II, 4a, p. 75.

If D is a rectangle, with "half-sides" a, b, $a \geq b$, as in the figure, it may be shown that $f_D(\rho)$ is given by the following expression:



$$f_D(\rho) = \begin{cases} 1 & 0 \leq \rho \leq b \\ \frac{1}{2\pi} \sin^{-1} b/\rho & b \leq \rho \leq a \\ \frac{1}{2\pi}\left[\sin^{-1} b/\rho - \sin^{-1} a/\rho\right] & a \leq \rho \leq \sqrt{a^2+b^2} \\ 0 & \rho \geq \sqrt{a^2+b^2} . \end{cases}$$

Again, if D is an annular band, situated as shown, of width d, central angle θ and inner radius A then

$$f_D(\rho) = \begin{cases} 0 & 0 \leq \rho \leq A \\ \theta/2\pi & A \leq \rho \leq A+d \\ 0 & \rho > A+d . \end{cases}$$

For certain other domains D of simple shape, the function $f_D(\rho)$ can be given explicitly. Of course, it goes without saying that if D is irregular it will in general be impossible to give an explicit formula for $f_D(\rho)$. In this connection, it is worthwhile to mention that the function $f_D(\rho)$ is <u>additive</u>, i.e. if D_1, D_2 are two disjoint domains, and $D_1 \cup D_2$ their union, then $f_{D_1 \cup D_2}(\rho) = f_{D_1}(\rho) + f_{D_2}(\rho)$. In practice then, a given domain D of irregular shape may be approximated by the union of non-overlapping domains, $D_1, D_2, \ldots, D_n$ each of which is of a simple regular shape. Then $f_{D_1}(\rho) + \cdots + f_{D_n}(\rho)$ is an approximation to $f_D(\rho)$.
Remark: Another interesting property of $f_D(\rho)$ is the following: If $A(D)$ denotes the area of the domain D then

$$A(D) = 2\pi \int_0^\infty \rho f_D(\rho)\, d\rho .$$

II). If D is a circle centered at the origin, we have seen that the integral $\psi(\xi; k)$ may be explicitly evaluated. In the case of the annular domain, also, it is possible to carry out the integration in (4), using the expression given above for $f_D(\rho)$. However, if D is a rectangle, we already run up against a rather formidable analytic problem.

If we substitute in (4) the expression for $f_D(\rho)$ given above for the rectangle, and simplify, we obtain expressions involving integrals of the form

$$(5) \qquad \int_{\sqrt{A^2+\xi^2}}^{\sqrt{B^2+\xi^2}} \sin^{-1}\frac{c}{\sqrt{r^2-\xi^2}}\, e^{-ikr}\, dr \qquad --$$

in which $c \le A < B$.

This type of integral seems very difficult to evaluate. One potentially successful method of handling it may be the following.

Let $I(\xi, k)$ denote the integral

$$I(\xi, k) = \int_A^B f(\sqrt{r^2-\xi^2})\, e^{-ikr}\, dr$$

where f is an arbitrary function, subject only to the condition that $\int_A^B f(x)\,dx$ exists. If we set $s = \sqrt{r^2-\xi^2}$ we obtain:

$$(6) \quad I(\xi, k) = \int_{\sqrt{A^2-\xi^2}}^{\sqrt{B^2-\xi^2}} f(s)\,\frac{e^{-ik\sqrt{s^2+\xi^2}}}{\sqrt{s^2+\xi^2}}\, s\, ds$$

From Campbell and Foster, as above, Part II, we have:

$$(7) \quad \frac{e^{-ik\sqrt{s^2+\xi^2}}}{\sqrt{s^2+\xi^2}} = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0\left[\xi\sqrt{q^2-k^2}\right] e^{-isq}\, dq\,.$$

Substituting this into the expression (6), and formally inverting the two integrals gives:

$$(8) \quad I = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0\left(\xi\sqrt{q^2-k^2}\right) dq \int_{\sqrt{A^2-\xi^2}}^{\sqrt{B^2-\xi^2}} s\, f(s)\, e^{-iqs}\, ds\,.$$

Now $\int s\, f(s) e^{-iqs}\, ds = -1/i\ d/dq\ I(0;q)$ as is evident from the definition of I. Thus (8) becomes:

$$(9)\quad I(\zeta;k) = -\frac{1}{\pi i}\int_{-\infty}^{\infty} K_0\left[\zeta\sqrt{q^2-k^2}\right] d/dq\ I(0;q)\, dq.$$

Integrating by parts we obtain:

$$(10)\quad I(\zeta;k) = -\frac{1}{\pi i}\left[K_0(\zeta\sqrt{q^2-k^2})I(0;q)\right]_{-\infty}^{\infty} +$$

$$+\frac{1}{\pi i}\int_{-\infty}^{\infty} I(0;q)\ d/dq\ K_0(\zeta\sqrt{q^2-k^2})\, dq.$$

From Higher Transcendental Functions, Bateman Manuscript Project, Erdelyi et al; Vol. II, p. 79, we have:

$$d/dz\ K_0(z) = -K_{-1}(z).$$

Using this in (10) yields:

$$(11)\quad I(\zeta;k) = -\frac{1}{\pi i}\left[K_0(\zeta\sqrt{q^2-k^2})I(0;q)\right]_{-\infty}^{\infty} -$$

$$-\frac{\zeta}{\pi i}\int_{-\infty}^{\infty} I(0;q)\,\frac{K_{-1}(\zeta\sqrt{q^2-k^2})}{\sqrt{q^2-k^2}}\, q\, dq.$$

Since (Higher Transc. Functions, ibid., p. 23) $K_0(\infty) = 0$, and since $I(0;q)$ remains bounded as $q \to \infty$ we see that

$$\left[K_0(\zeta\sqrt{q^2-k^2})\,I(0;q)\right]_{q=\infty} = 0$$

and (11) becomes:

$$(12)\quad I(\zeta;k) = \frac{1}{\pi i}\left[K_0(\zeta\sqrt{q^2-k^2})\,I(0;q)\right]_{q=-\infty}$$

$$-\frac{\zeta}{\pi i}\int_{-\infty}^{\infty} I(0;q)\,\frac{K_{-1}(\zeta\sqrt{q^2-k^2})}{\sqrt{q^2-k^2}}\,q\,dq\,.$$

This last equation is our desired result. Its signigicance is that it relates the evaluation of $I(\zeta;k)$ to that of $I(0;q)$.

If for $f(r)$ we take $f_D(r)$ for a given domain D, then $I(\zeta,k)$ is just $\psi(\zeta;k)$ and (12) specializes to:

$$(13)\quad \psi(\zeta;k) = \frac{1}{\pi i}\left[K_0(\zeta\sqrt{q^2-k^2})\,\psi(0;q)\right]_{q=-\infty}$$

$$-\frac{\zeta}{\pi i}\int_{-\infty}^{\infty} \psi(0;q)\,\frac{K_{-1}(\zeta\sqrt{q^2-k^2})}{\sqrt{q^2-k^2}}\,q\,dq\,.$$

Thus the velocity potential at the field point $(0,0,\zeta)$ with propagation constant k may be expressed, by this last relation, in terms of the potential for $\zeta = 0$.

If for $f(r)$ we take $\sin^{-1} c/r$, as in (5), we see that the evaluation of (5) reduces to the evaluation of the integral

$$\int_A^B \sin^{-1}\frac{c}{r}\,e^{-ikr}\,dr$$

which is the form (5) takes for $\zeta = 0$. This latter may be transformed (we do not give the details here) into an integral of the form

$\int \hat{f}(\gamma)\, H(\gamma)\, d\gamma$ where $\hat{f}$ is the Fourier transform of $\sin^{-1} s$, and H is the kernel function for an integral transform of the type of the Bessel transform.

Evidently, these formal manipulations need to be supplemented by much more careful analysis in order to reach a satisfactory end-result.

Part IV

The Basic Equations

The starting point is again the basic integral representation of the velocity potential ψ due to a source distribution on an infinite rigid plane S:

$$(1) \qquad \psi = -\frac{1}{2\pi} \int_S \frac{\partial\psi}{\partial n} \frac{e^{-ikr}}{r}\, dS$$

ψ .=. velocity potential

S .=. integration extends over the entire plane

r .=. distance, source point to field point

$$(2) \qquad w = \frac{\partial\psi}{\partial n}$$.=. normal velocity distribution on the plane.

$$(3) \qquad p = -i\rho\omega\psi$$.=. acoustic pressure.

$\dfrac{e^{-ikr}}{r}$.=. Green's Function for scalar Helmholtz equation, for the plane.

For a sound radiation field having a single frequency component ω, ψ is proportional to acoustic pressure. It will be convenient to deal with $\psi(P, k)$, the velocity potential. This is a function of position P of the field point and the propagation constant $k = \omega/c = 2\pi/\lambda$. We thus avoid

carrying along some constants that add nothing to the discussion at hand. If $w(x,y)$ is the velocity at every point in the plane of a baffle and pistons and if (ξ, η) are the coordinates of an arbitrary field point P on the plane then

$$(4)\quad \psi(\xi,\eta;k) = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} w(x,y)\,\frac{e^{-ikr}}{r}\,dx\,dy$$

where

$$(5)\quad r^2 = (x-\xi)^2 + (y-\eta)^2 + \zeta^2.$$

Here we take $\zeta = 0$, considering only points on the plane $z = 0$. In this special case it is convenient to introduce polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

and consider ψ at the origin $\xi = 0$, $\eta = 0$. We may then write,

$$(6)\quad \psi(k) = \psi(0,0;k) = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} w(r,\theta)\,e^{-ikr}\,dr\,d\theta.$$

Here, we have expressed the piston velocity distribution $w(r,\theta)$ in polar coordinates with the field point of interest as origin. This entails no real loss in generality.

If we wish to get the position of the field point back into the picture we may write,

$$(7)\quad \psi(\xi,\eta;k) = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} w(r,\theta,\xi,\eta)\,e^{-ikr}\,dr\,d\theta$$

where,

$w(r,\theta,\xi,\eta)$.=. velocity distribution in polar coordinates with the field point (ξ, η) as origin.

However, for the moment we do not carry the excess notational baggage of equation (7) and use equation (6) instead. Finally let us split equation (6) into two equations:

$$(8) \quad V(r) = \frac{1}{2\pi} \int_0^{2\pi} w(r,\theta)\, d\theta$$

and

$$(9) \quad \psi(k) = -\int_0^{\infty} V(r) e^{-ikr}\, dr \, .$$

These equations are basic to what follows. In general $w(r,\theta)$ is complex and the modules of w is the magnitude of the piston velocity. The argument of w is the phase.

Some General Considerations.

Before continuing it is important to observe that the area of any single baffled piston may be idealized to be an area over which $w(r,\theta)$ is a constant. Thus one may write

$$(10) \quad w(r,\theta) = \sum_{i=1}^{N} a_i \, c_i(r,\theta)$$

where

a_i .=. complex number giving phase and velocity amplitude of piston i

$c_i(r,\theta)$.=. characteristic function of piston i

$$c_i(r,\theta) = \begin{cases} 1 \text{ on piston } i \\ 0 \text{ everywhere else.} \end{cases}$$

This leads to the consideration of the triple of equations

$$(11) \quad V_i(r) = \frac{1}{2\pi} \int_0^{2\pi} c_i(r,\theta)\, d\theta$$

$$(12) \quad \psi_i(k) = -\int_0^{\infty} V_i(r) e^{-ikr}\, dr$$

and

$$(13)\quad \psi(k) = \sum_{i=1}^{N} a_i \psi_i(k).$$

The study of an analog technique can now be broken down into the problem of generating $c_i(r,\theta)$ and calculating each of the expressions $v_i(r)$, $\psi_i(k)$, $\psi(k)$ and finally the modulus of $\psi(k)$ which is proportional to the magnitude of the sound pressure.

A Function Generator for $c_i(r,\theta)$

The simple form of the function $c_i(r,\theta)$ permits it to be generated by an optical device as follows. In front of a cathode ray tube screen is placed a mask which is scaled to represent the plane baffle in the neighborhood of the array. The mask is transparent in the region corresponding to piston i and is opaque everywhere else.

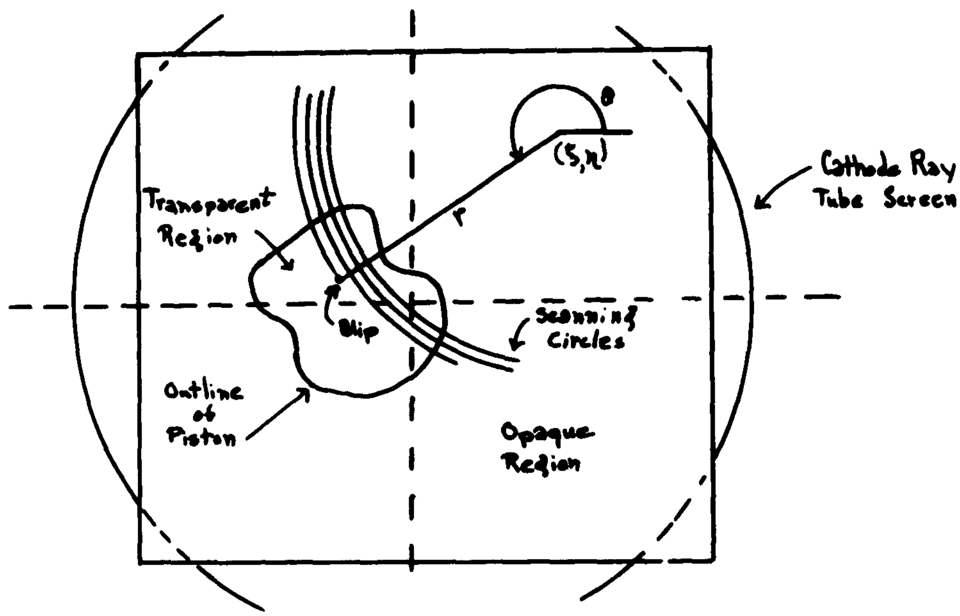


Figure 1

The null-deflection point of the CRT (cathode ray tube) is set at a point on the screen which corresponds to the field point (ξ, η) on the mask. The blip of the CRT is set to scan in concentric circles about this point. If the blip on the CRT screen is started at the null point and circles about it in circles of gradually increasing radius the entire mask will be covered in a relatively "dense" fashion. See Figure 1.

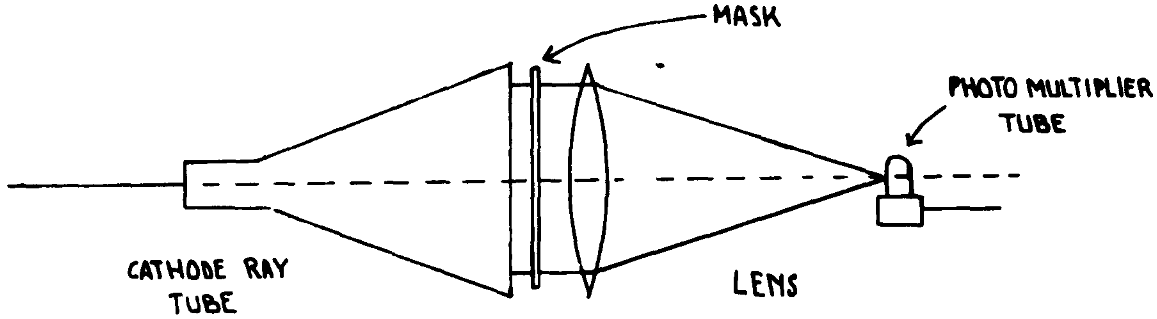


Figure 2

Light passing through the transparent region of the mask is focused by a lens on a photo multiplier tube. The signal from the photo multipler tube will be a parametric representation of $c_i(r, \theta)$ as follows:

(14) $\theta = 2\pi\nu t$, where ν .=. frequency of circular scan

(15) $r = \Delta r\, \nu t$

where Δr is the increment of radius with each scan cycle. $\Delta r \ll 1$.

Accuracy of the $c_i(r, \theta)$ Generator.

In order to measure the accuracy of the proposed $c_i(r, \theta)$ generator a manual calculation was made following the method of the analog generator for a nine piston array for which the sound pressure magnitude is available in the Sherman and Kass report. (Sherman, Charles H. and Kass, Donald F., "Near Field Sound Pressure of Arrays of Pistons". U. S. Navy Underwater Sound Laboratory, New London, Connecticut, 1961.)

Calculations of $c(r,\theta)$ were made for thirteen field points or calculation stations located along the axis of centers of the middle pistons. (See Figure 20). Reference circles were constructed corresponding to the circular scan of the CRT. The increment of radius was chosen so that the array was covered in 75 circles.

Since all the pistons were taken to be vibrating at the same frequency, phase, and amplitude it was unnecessary to calculate $c_i(r,\theta)$ for each piston separately. All pistons were considered at once giving $c(r,\theta)$. The symmetry of the array permitted the consideration of only the pistons (and halves of pistons) on one side of the axis of field points.

From the values of $c(r,\theta)$ thus determined the sound pressure amplitude was calculated by numerical methods.

Figures (7) to (19) show $v(r)$ in graphical form for each calculation station P_i . Figure (20) gives the comparison of the two methods of calculation in the form of a graph of the sound pressure amplitude vs. length along the axis of field points. The array is characterized by $ka = \pi/2$, using the notation of Sherman and Kass, where a is the piston radius.

It may be seen that the discrepancy is of the order of 1%. Clearly higher accuracy will be obtained by using a smaller increment of radius and more field points.

<u>Computation Device for $v_i(r)$, $\mathrm{Re}\,\psi_i(t_k)$ and $\mathrm{Im}\,\psi_i(t_k)$</u>.

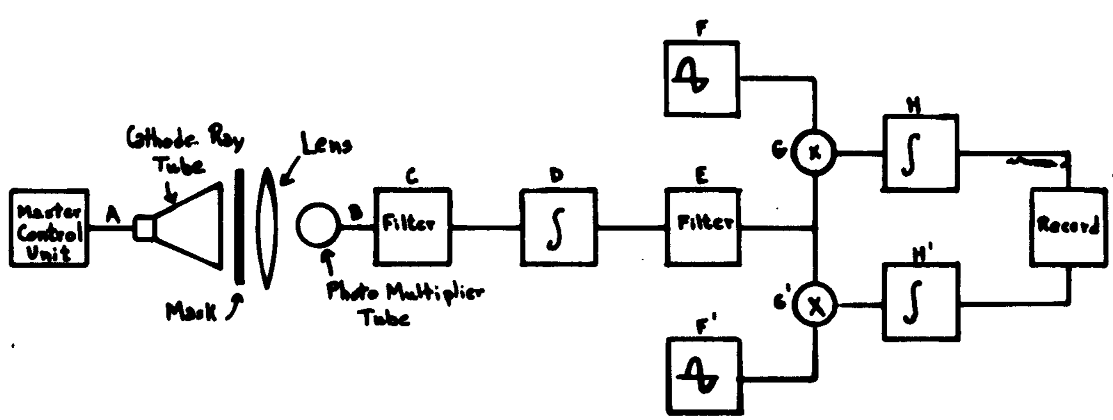
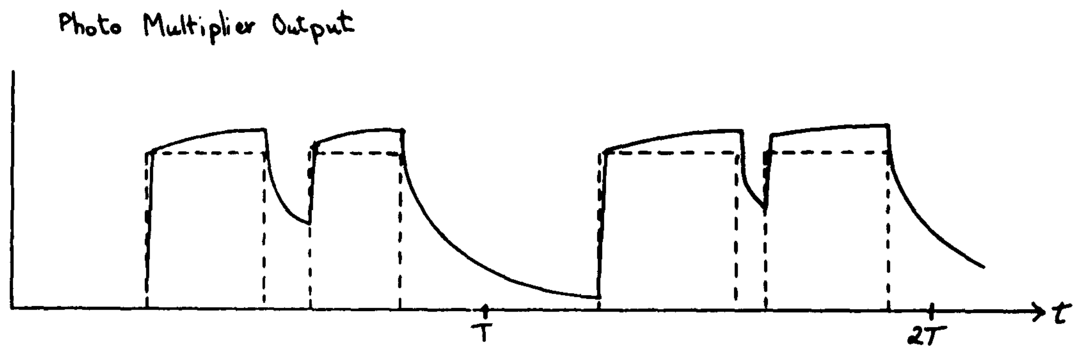


Figure 3

Figure 3 shows the suggested device for computing $v_i(r)$, and from that, $\mathrm{Re}\ \psi_i(t)$ and $\mathrm{Im}\ \psi_i(t)$.

A. Scanning signal



______ Actual output with afterglow effects for high speed operation.

\- - - - - - Idealized output with no afterglow effects.

B. Photo multiplier signal

C. The switching filter gives a rectangular pulse output which closely approximates the ideal photo multiplier output (with no afterglow effects) for a high cut-off threshold.

D. The integrator D, which is set to dump once each period, gives $\int_0^T c_i(r,\theta)dt$. Taking r to be a constant over one period, this is mathematically equivalent to $\frac{1}{2\pi}\int_0^{2\pi} c_i(r,\theta)\,d\theta$ by equation (14). This is within a multiplicative constant of $v_i(r)$. Thus the output of the integrator is a sequence of pulses each one of which starts from zero and climbs to a maximum value proportional to $v_i(r)$. See Figure 5.

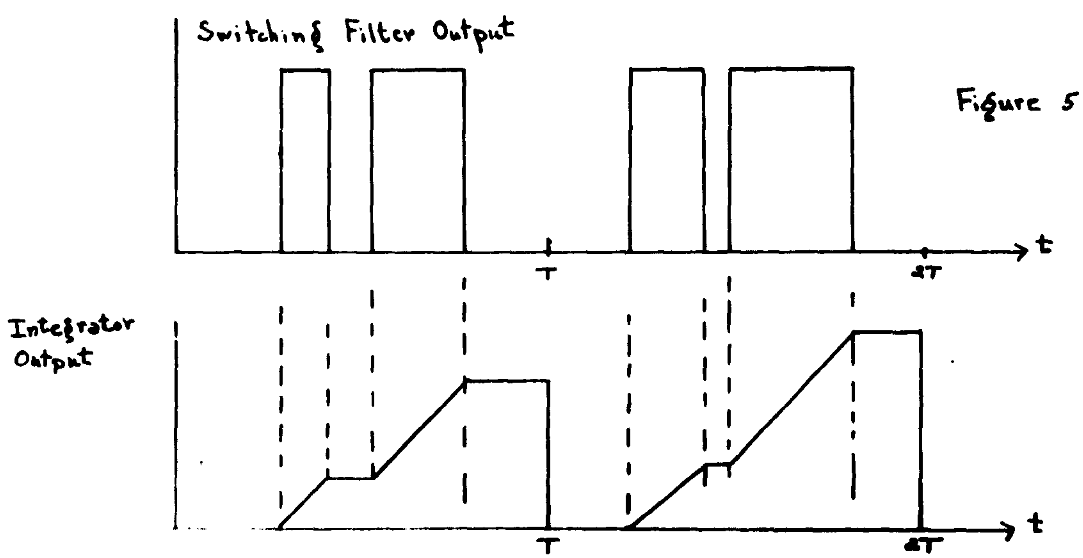


Figure 5

E. The low pass filter gives the envelope of the pulse output of the integrator, i.e. a continuous reading of $V_i(r)$. See Figure 6.

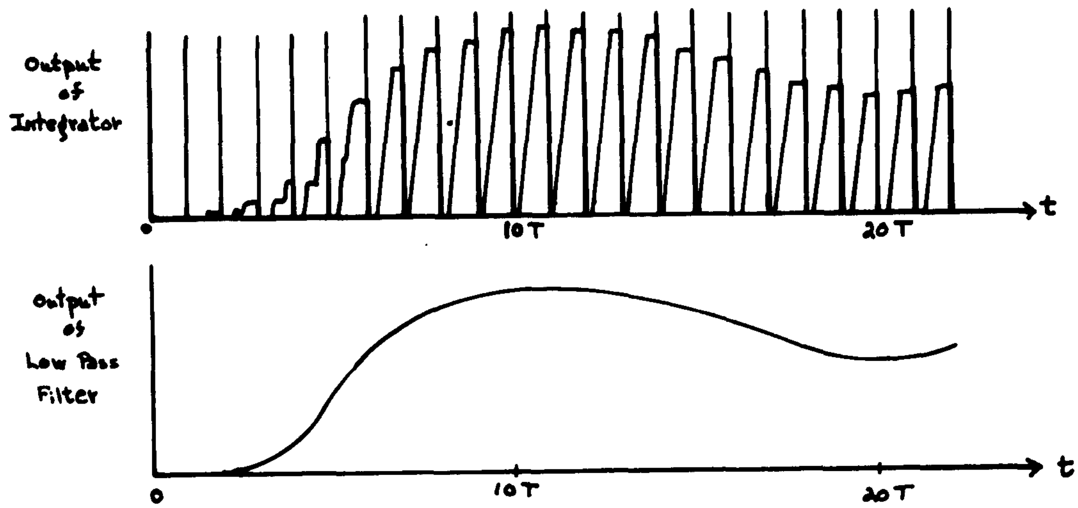


Figure 6

In Figure 6 the time scale is over many periods so that r may not be considered constant but is related to t by equation (15).

F and F^1. The oscillators F and F^1 give sinusoidal outputs as a function of kr . The phase of F is ahead of the phase of F^1 by $\pi/2$.

G and G^1. The multipliers G and G^1 give $V_i(r)\,\mathrm{Cos}\,kr$ and $-V_i(r)\,\mathrm{Sin}\,kr$ respectively.

H and H^1. The integrators H and H^1 give $\int_0^t V_i(r)\,\mathrm{Cos}\,kr\,dt$ and $-\int_0^t V_i(r)\,\mathrm{Sin}\,kr\,dt$ respectively. This is equivalent to $1/\Delta r\nu \int_0^r V_i(r)\,\mathrm{Cos}\,kr\,dr$ and $-1/\Delta r\nu \int_0^r V_i(r)\,\mathrm{Sin}\,kr\,dr$. It will be noted that for $r > r_{max}$. $V_i(r) = 0$.

Thus the upper limit of integration is equivalent to ∞ after r reaches r_{max} . Thus the outputs of the integrators are signals whose final values are proportional to $\mathrm{Re}\ \psi_i(k)$ and $\mathrm{Im}\ \psi_i(k)$ respectively. These quantities are stored in a recording device.

Automatizing the Computer.

The system outlined above gives the real and imaginary parts of $\psi_i(k)$ evaluated at one field point corresponding to the null-deflection point of the CRT on the mask. The CRT scanning control may be programmed to cease the circular scan after the radius has increased to a value greater than r_{max}, relocate the null-deflection point to some point neighboring the first, and start the circular scan at $r = 0$ again. The null point advancing device can be set to cover the entire mask with a lattice of field points.

The final stage integrators (H and H^1 in Figure 3) must be set to dump at the completion of computation for each field point. The output of these integrators will then be a sequence of pulses whose maximum (= cut-off) value corresponds to $\mathcal{R}e\ \psi_i(k; \xi, \eta)$ and $\mathcal{I}m\ \psi_i(k, \xi, \eta)$ for each field point (ξ, η) .

It is clear that the recording of ψ_i takes place several orders of magnitude more slowly than the circular sweep of the CRT. Thus the question arises: How long will the complete coverage of the mask take?

This problem has not been studied in detail but a brief inspection suggests the following estimate. A standard short persistance oscilloscope coating retains 10% of its initial glow 2 milliseconds after a sharp cutoff of the beam, and 1% after 3 milliseconds. (Czeck, J. The Cathode Ray Oscilloscope. Interscience Publishers, Inc., N. Y., 1957, p. 21.) If the filter which minimizes afterglow effects is well designed it should be possible to operate the circular scan at up to 200 cps. About 200 cycles should suffice for an accurate coverage of the piston region so that one field point can be calculated per second. A lattice of 25 x 25 field points could then be covered in less than eleven minutes. Recent improvements in shortening afterglow time may permit this figure to be reduced.

Systems for Computing the Sound Pressure Amplitude from $\mathcal{R}e\ \psi_i$ and $\mathcal{I}m\ \psi_i$ Data.

There are a large number of ways of computing the sound pressure amplitude, $|p|$, from the values of $\mathcal{R}e\ \psi_i$ and $\mathcal{I}m\ \psi_i$. Many of these are now under study for feasibility and practicality. A few of the $\mathcal{R}e\ \psi_i$ and $\mathcal{I}m\ \psi_i$ recording devices and computation devices are outlined below.

1. Recording $\mathrm{Re}\,\psi_i$ and $\mathrm{Im}\,\psi_i$ as distinct quantities for each field point with digital calculation of $|p|$.
2. Recording $\mathrm{Re}\,\psi_i$ and $\mathrm{Im}\,\psi_i$ as continuous level curves corresponding to lines of field points with graphical or digital calculation of $|p|$.
3. Recording $\mathrm{Re}\,\psi_i$ and $\mathrm{Im}\,\psi_i$ as continuous darkening of a photographic film corresponding to the CRT mask, i.e. corresponding to the region of the array. The light source which exposes the film could be restricted so as to expose a small region of the film corresponding to the field point, and to advance as the field point does. The darkening of the film would then represent the $\mathrm{Re}\,\psi_i$ or the $\mathrm{Im}\,\psi_i$ field in the neighborhood of the array. Optical devices using these films could perform the multiplication and summation in equation (13) for the real and imaginary parts separately and then give the modulus $|\psi|$ by vector addition of cross-polarized components. Thus $|p|$ can be obtained by entirely analog means. The optical system, however, diminishes rapidly in accuracy as the number of sets of pistons with the same amplitude, frequency and phase increase. The optical system should prove to be useful for accurate data on simple arrays and qualitative data on complex arrays. The latter should be of sufficient accuracy to identify and locate pressure peaks.

Conclusions and Recommendations

This report has presented analytical ideas and calculations so far accomplished. It has also presented preliminary ideas for analog pressure distribution calculation and display. It is felt that both analogs and mathematical analysis have much to offer in gaining an understanding of various features of the near sound field of transducers and arrays. The analog approach should allow a rapid semi-quantitative assessment of the value of various design ideas, with no limitations on transducer shapes and distribution. The analytic approach will continue to give the necessary deep insight into the fine details of array geometries that are sufficiently simple for their representation in convenient mathematical expressions. The development of analytic approximation techniques should be continued, the objective being

useful engineering formulae. Another important phase of future study should be a study of the applicability of x-ray crystallographic mathematics for sound array problems. This should teach us tricks and limitations.

APPENDIX

I. Analysis of $\delta_s(k - x)$

In the fourth section of Part I we were concerned with the evaluation of

$$(1)\quad I_2 = 2\int_0^\infty \frac{\pi i}{2\pi\sqrt{\lambda^2+\mu^2+\nu^2}}\,\delta_s\left(k - 2\pi\sqrt{\lambda^2+\mu^2+\nu^2}\right) d\nu$$

where previously $\delta_s(k - x)$ was defined as

$$(2)\quad \delta_s(k - x) = 2/\pi \int_0^\infty \sin(ky)\ \sin(xy)\, dy.$$

Substitution of (2) in (1) gives

$$(3)\quad I_2 = i\ 2/\pi \iint_0^\infty \frac{\sin(ky)\ \sin\left(2\pi\sqrt{\lambda^2+\mu^2+\nu^2}\ y\right)}{\sqrt{\lambda^2+\mu^2+\nu^2}}\, d\nu\, dy.$$

Let $\quad x = 2\pi\sqrt{\lambda^2+\mu^2+\nu^2} \qquad\qquad x^2 = (2\pi)^2(\lambda^2+\mu^2+\nu^2)$

$$dx = \frac{(2\pi)^2\,\nu\, d\nu}{2\pi\sqrt{\lambda^2+\mu^2+\nu^2}} \qquad\qquad \nu^2 = \frac{x^2-(2\pi)^2(\lambda^2+\mu^2)}{(2\pi)^2}$$

$$x\,dx = (2\pi)^2\,\nu\, d\nu \qquad\qquad \nu = \frac{\sqrt{x^2 - S_2}}{2\pi}$$

$$\frac{x\,dx}{(2\pi)^2\nu} = d\nu$$

(4)

$$I_2 = 2i/\pi \int_0^\infty \int_{\sqrt{S_2}}^\infty \frac{\sin(ky)\ \sin(xy)}{\frac{x}{2\pi}}\ \frac{x\,dx}{(2\pi)^2\frac{\sqrt{x^2-S_2}}{2\pi}}\ dy$$

$$I_2 = 2i/\pi \int_0^\infty \sin(ky)\,dy \int_{\sqrt{S_2}}^\infty \frac{\sin(xy)}{\sqrt{x^2-S_2}}\ dx$$

Reference: Gröbner Wolfgang und Hofreiter Nikolaus, "Integraltafel-Zweiter Teil, Bestimmte Integrale", Springer-Verlag, 1950, p. 129, No. 77c.

$$\int_1^\infty \frac{\sin zx}{\sqrt{x^2-1}}\,dx = \pi/2\ J_0(z) \qquad z>0.$$

We have

$$A=\int_{\sqrt{s_2}}^\infty \frac{\sin xy}{\sqrt{x^2-s_2}}\,dx = 1/\sqrt{s_2}\int_{\sqrt{s_2}}^\infty \frac{\sin xy}{\sqrt{x^2/s_2-1}}\,dx.$$

Let $\xi = \dfrac{x}{+\sqrt{s_2}}$ then $\begin{cases}\xi=1 & \text{when } x=\sqrt{s_2}\\ \xi=\infty & \text{when } x=\infty\end{cases}$

$$d\xi=\frac{dx}{\sqrt{s_2}}, \qquad z=\sqrt{s_2}\,y>0 \quad \text{for} \quad y>0.$$

Then

$$A=\int_1^\infty \frac{\sin z\xi\,d\xi}{\sqrt{\xi^2-1}} = \frac{\pi}{2}J_0(z)=\frac{\pi}{2}J_0(\sqrt{s_2}\,y).$$

Substituting this now into Equation (4)

$$I_2 = i\int_0^\infty \sin(ky)\,J_0(\sqrt{s_2}\,y)\,dy.$$

From Oberhettinger p. 165 § 15

$$I_2=\begin{cases}0 & k<\sqrt{s_2}\\ \dfrac{i}{\sqrt{k^2-s_2}} & k>\sqrt{s_2}.\end{cases}$$

This agrees with the results that were obtained by merely treating δ_s as the Dirac delta function between the limits of $\sqrt{s_2}$ and ∞.

II. Summary of Formulas

1. $\psi_2(\xi,\eta;k)=\iint_{-\infty}^{\infty}\phi_2(\lambda,\mu;k)\,e^{2\pi i(\lambda\xi+\mu\eta)}\,d\lambda\,d\mu$

2. $\phi_2(\lambda,\mu;k)=\iint_{-\infty}^{\infty}K_2(\lambda,\mu,x,y;k)\,w(x,y)\,dx\,dy$

3. $K_2(\lambda,\mu,x,y;k) = -\dfrac{1}{\sqrt{s_2 - k^2}}\, e^{-2\pi i(\lambda x + \mu y)}$ where $s_2 = (2\pi)^2(\lambda^2+\mu^2)$, $k^2 \neq s_2$

4. $F_2(\xi,\eta;k) = |\psi_2|^2 = \psi_2(\xi,\eta;k)\,\psi_2^*(\xi,\eta;k)$ or

$$F_2(\xi,\eta;k) = \iint_{-\infty}^{\infty} \Phi_2(\lambda'',\mu'';k)\, e^{2\pi i(\lambda''\xi + \mu''\eta)}\, d\lambda''\, d\mu''$$

5.

$$\Phi_2(\lambda'',\mu'';k) = \iiiint\!\!\iint_{-\infty}^{\infty} K_2(\lambda,\mu,x,y;k)\, K_2^*(\lambda''-\lambda,\mu''-\mu;x',y';k)\, w(x,y)\, w^*(x',y')\, dx\, dy\, dx'\, dy'\, d\lambda\, d\mu$$

6.

$$\psi_3(\xi,\eta,\zeta;k) = \iiint_{-\infty}^{\infty} \phi_3(\lambda,\mu,\nu;k)\, e^{2\pi i(\lambda\xi+\mu\eta+\nu\zeta)}\, d\lambda\, d\mu\, d\nu$$

7.

$$\phi_3(\lambda,\mu,\nu;k) = \iint_{-\infty}^{\infty} K_3(\lambda,\mu,\nu;x,y;k)\, w(x,y)\, dx\, dy$$

8.

$$K_3(\lambda,\mu,\nu;k) = e^{-2\pi i(\lambda x+\mu y)}\left[\frac{2}{k^2 - s_3} + \frac{\pi i}{\sqrt{s_3}}\,\delta_s(k - \sqrt{s_3})\right]$$

where $s_3 = (2\pi)^2(\lambda^2+\mu^2+\nu^2)$.

9.

$$\delta_s(x - x') = 2/\pi \int_0^{\infty} \sin(xy)\, \sin(x'y)\, dy$$

10. (For a circular piston)

$$\phi_3(\lambda,\mu,\nu,k) = w_0 L_3 \frac{2\pi a}{\sqrt{s_2}}\, J_1(\sqrt{s_2}\, a)$$

where

$s_2 = (2\pi)^2(\lambda^2+\mu^2)$

$s_3 = (2\pi)^2(\lambda^2+\mu^2+\nu^2)$

$L_3 = \left[\dfrac{2}{k^2 - s_3} + \dfrac{\pi i}{\sqrt{s_3}}\,\delta_s(k - \sqrt{s_3})\right]$

$W_0 = W(x,y)$ when $x^2+y^2 \leq a^2$

11. (For a circular piston)

$$\psi_3(\xi,\eta,\zeta;k) = a w_0 \int_0^\infty \frac{e^{-\mu\zeta}}{\mu} J_0(r\rho)\, J_1(ra)\, dr$$

where $w_0 = w(x,y)$ when $x^2+y^2 \leq a^2$

$$\mu = \sqrt{r^2-k^2}$$

$$r = \sqrt{s_2}$$

$$\rho = \sqrt{\xi^2+\eta^2}$$

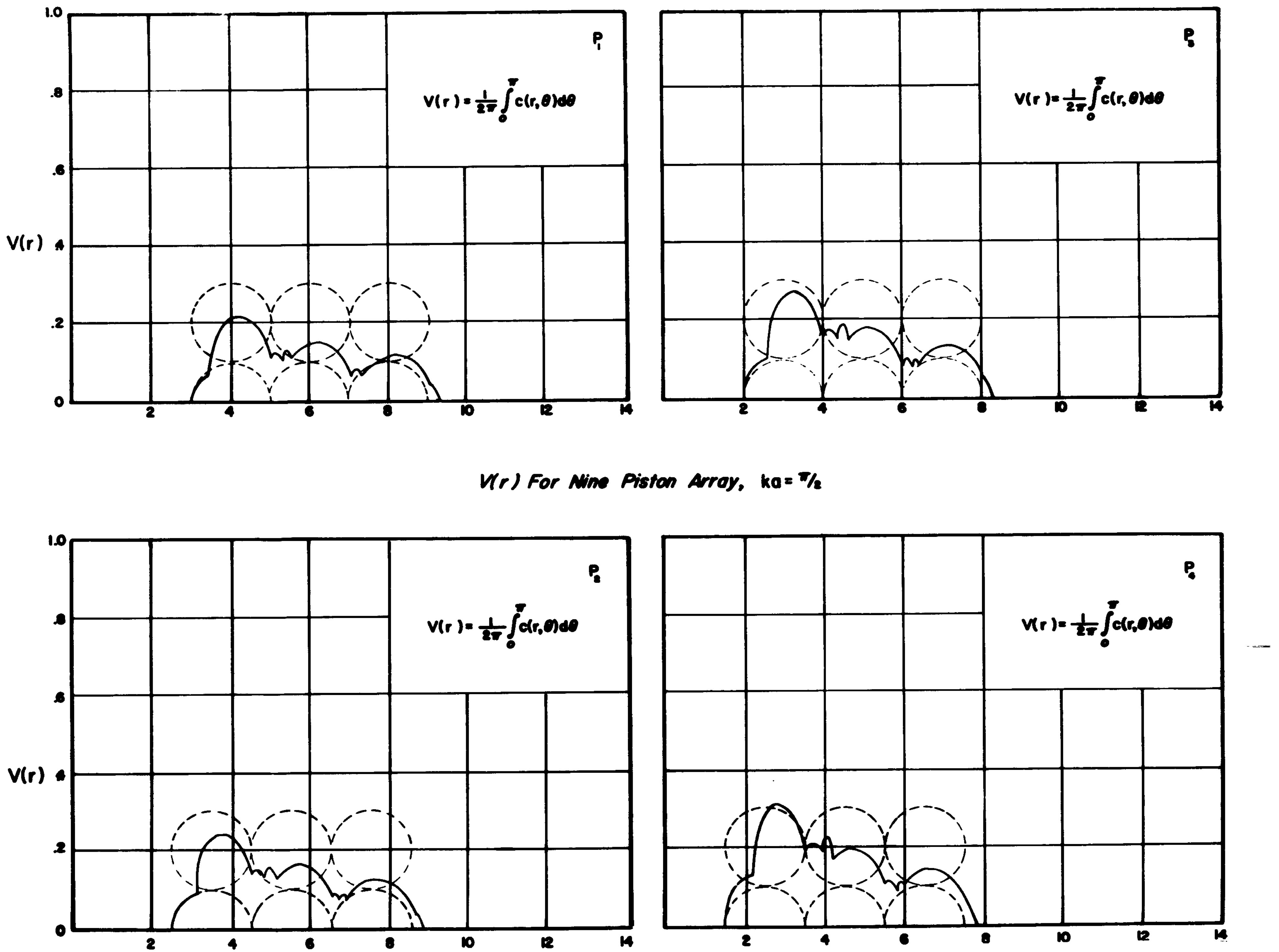
12. (For a rectangular piston)

$$\psi_3(\xi,\eta,\zeta;k) = \int_{-\infty}^{\infty}\int_{\eta-b}^{\eta+b} \frac{\sin(a\pi a\lambda)}{\pi^2\lambda} \cos(a\pi\lambda\xi) \cdot K_0\left[i\sqrt{k^2-(a\pi\lambda)^2}\sqrt{c^2+\zeta^2}\right] dc\, d\lambda$$

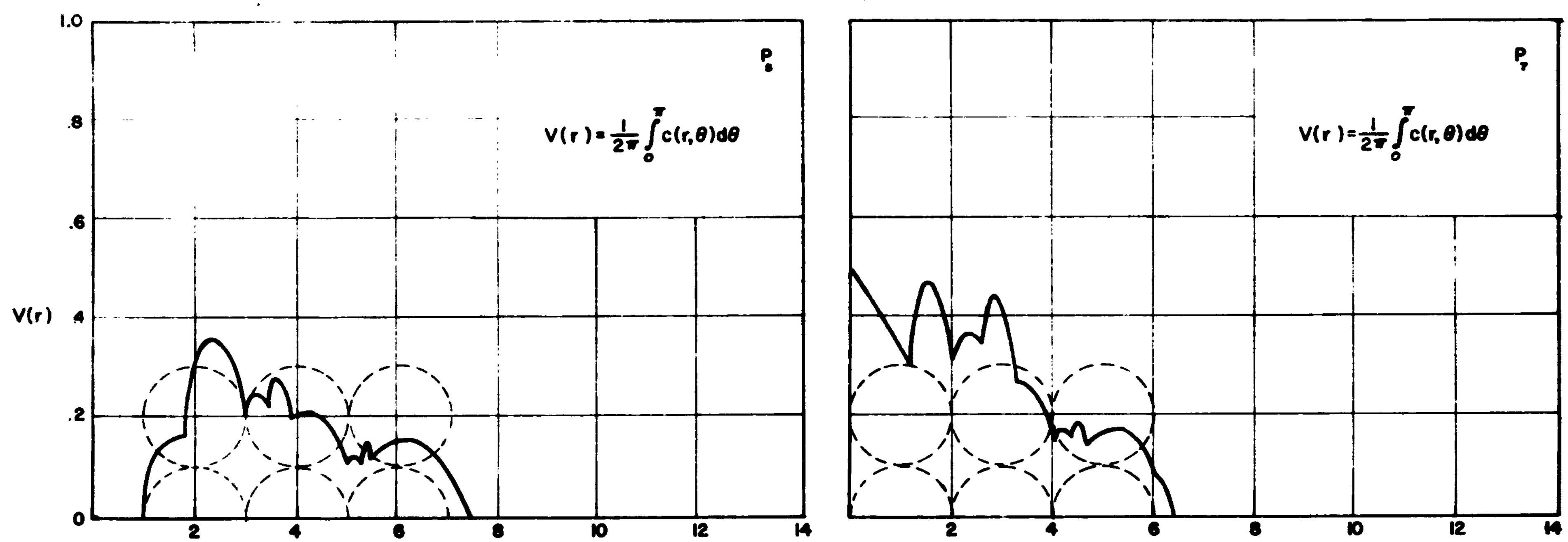
III. Graphs and Pressure Contours

1. Figures 7-19 show $v(r)$ calculated for the thirteen field points for the sample nine-piston array used in the study of the accuracy of the $c_i(r,\theta)$ generator. The horizontal scale is measured in units of a, the radius of the pistons, measured with respect to the particular field point as origin.

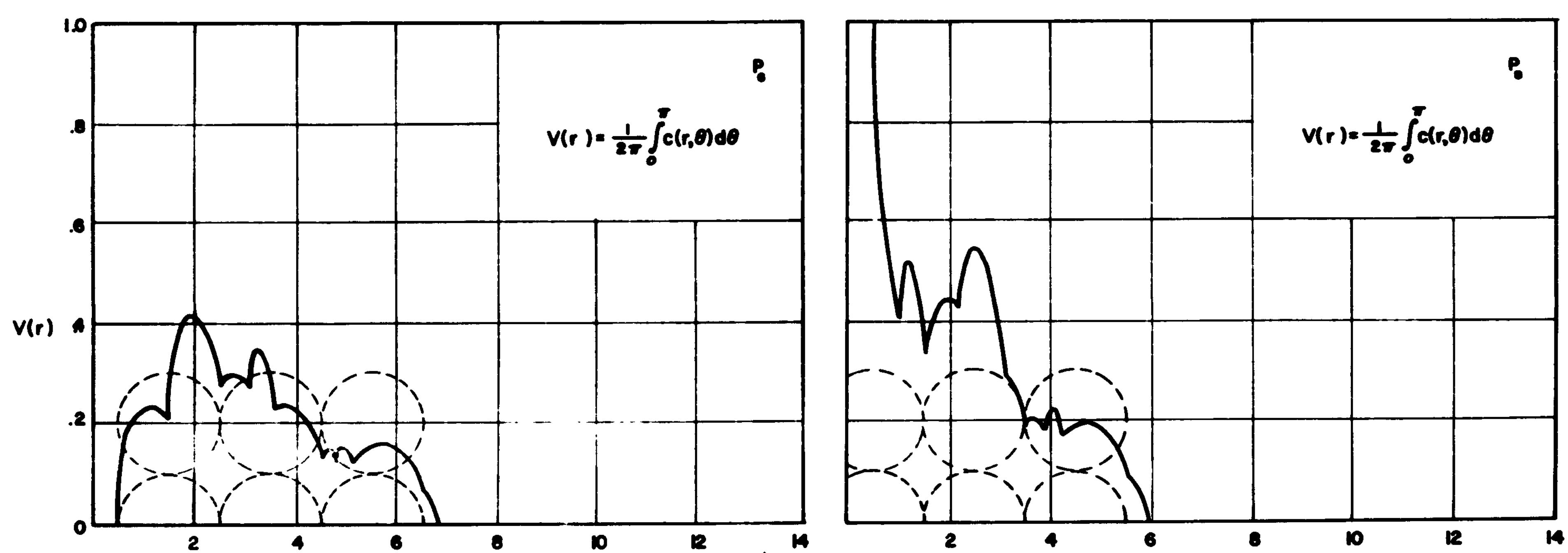
2. Figure 20 shows $\frac{|p|}{\rho c u}$ comparing the Sherman and Kass calculation with the Parke and Moran simulation of analog computation. The horizontal scale is measured in units of field point separation $= \frac{1}{2}a$.

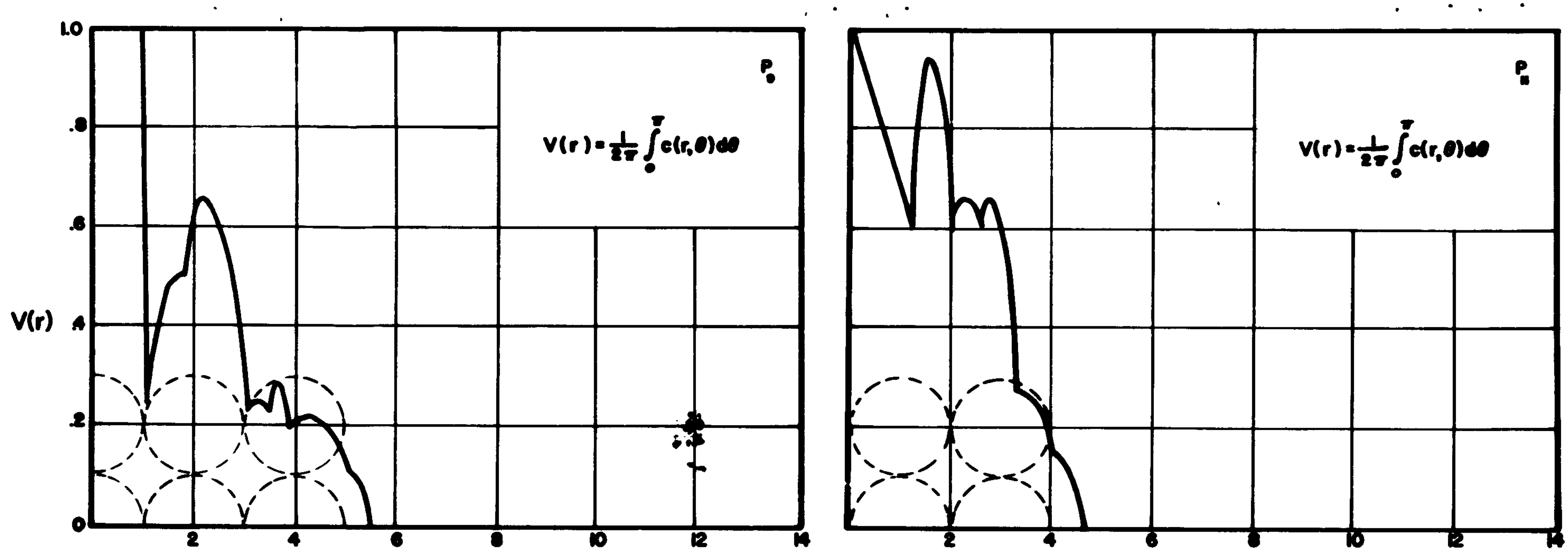


V(r) For Nine Piston Array, $ka = \pi/2$

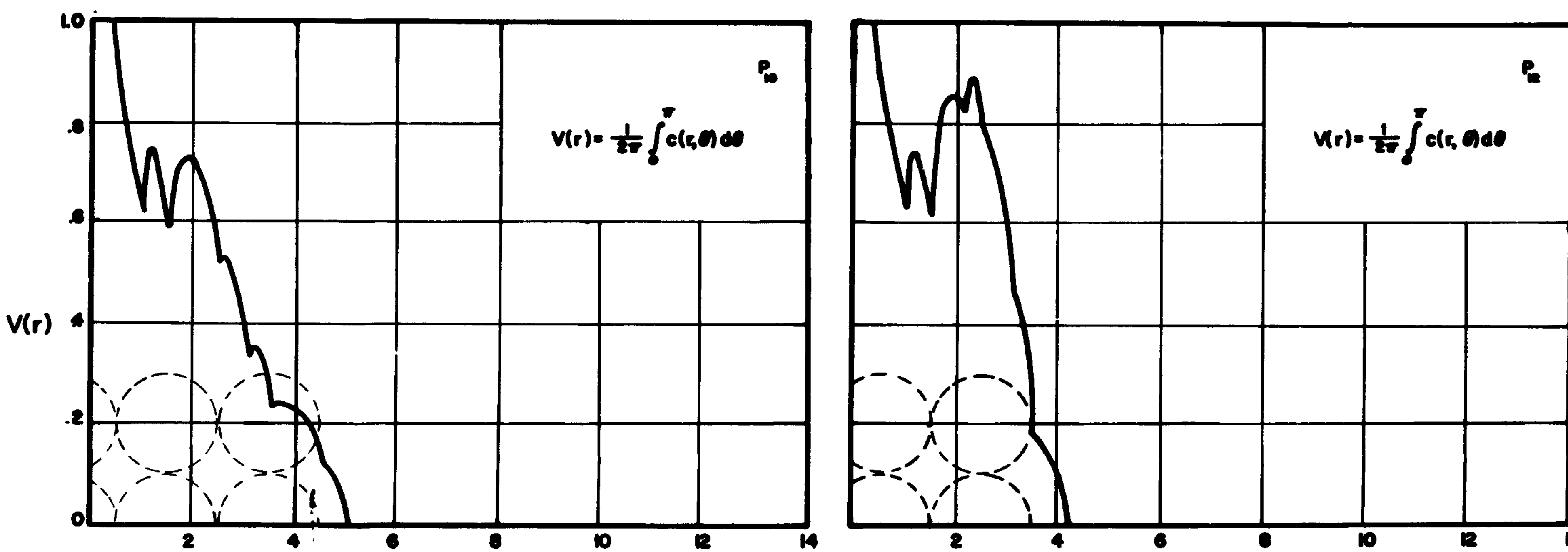


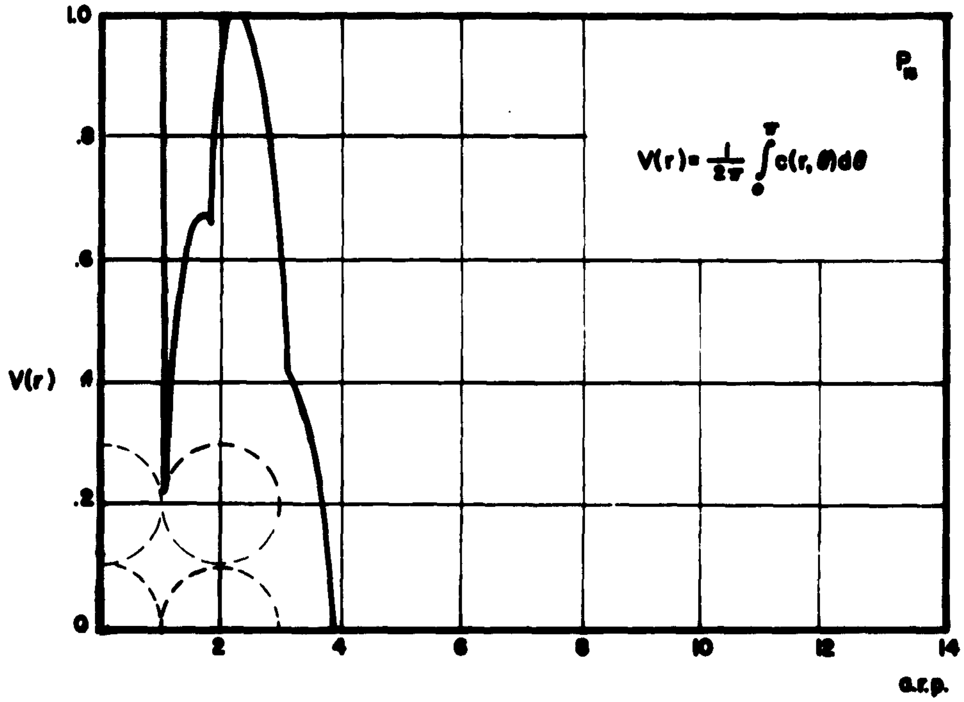
V(r) For Nine Piston Array, $ka = \pi/2$





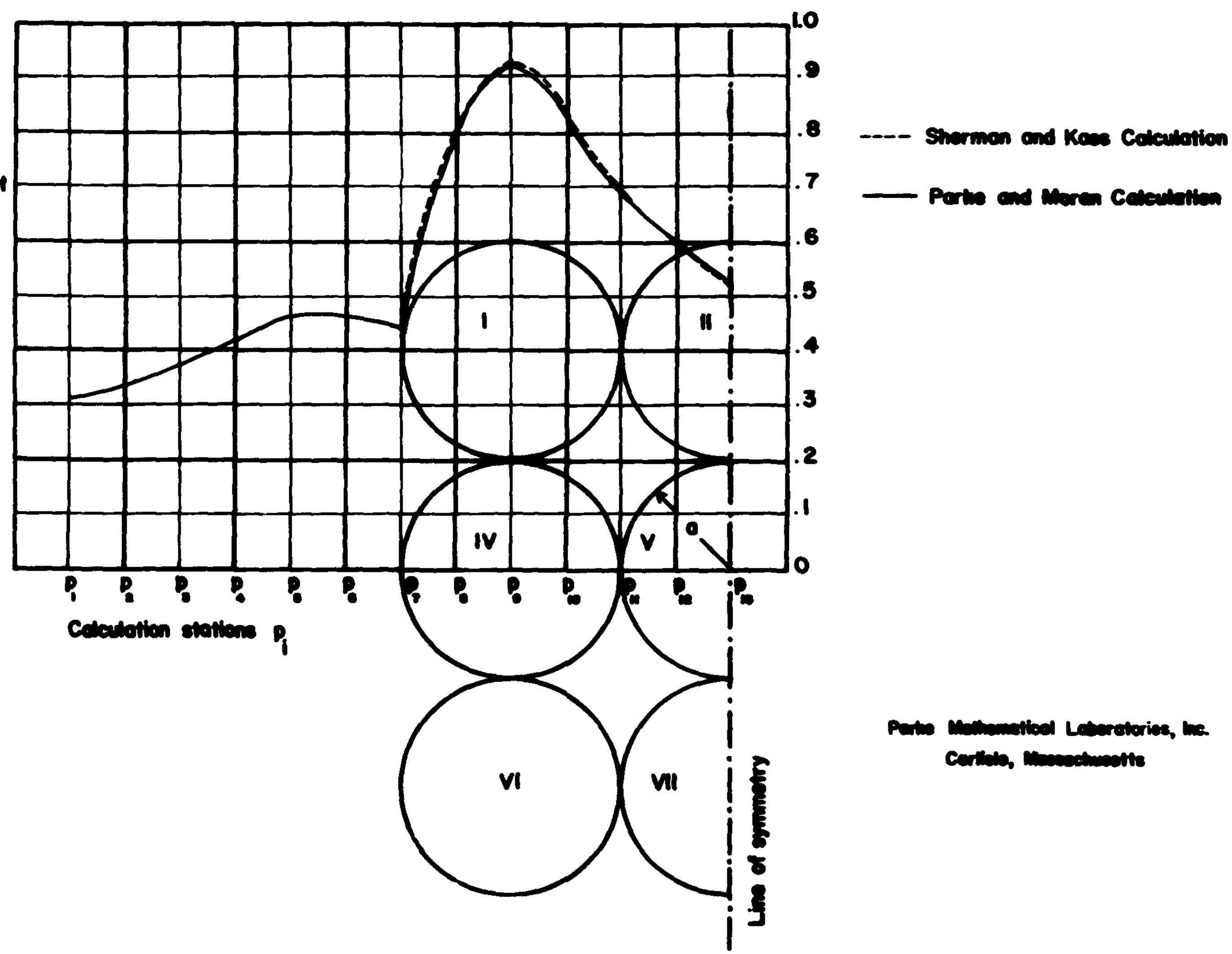
V(r) For Nine Piston Array, $ka = \pi/2$





V(r) For Nine Piston Array, ka= $\frac{5\pi}{2}$

Comparison of Two Methods of Calculation



Sound Pressure Amplitude ($|p|/\rho c u$) at Points $p_1 \cdots p_{13}$ for a 9 Element Array Composed of Half-Wave Length Diameter Pistons Beam Steering of 0°

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